Inverse z-transform

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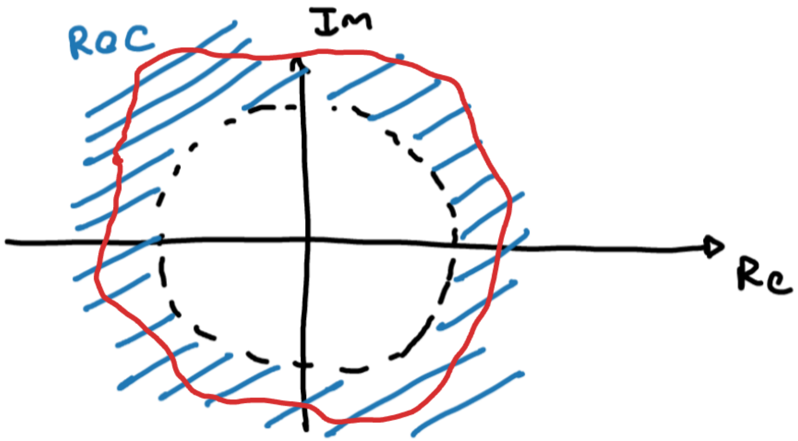
$$\begin{tikzcd} h[n] \arrow[r, bend left, "\mathcal{Z}\{\cdot\}"] & H(z) \arrow[l, bend left, "\mathcal{Z}^{-1}\{\cdot\}"] \end{tikzcd}$$

Inverse z-transform

$$\boxed{x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \, \mathrm{d}z}$$



Partial fraction expansion intuition

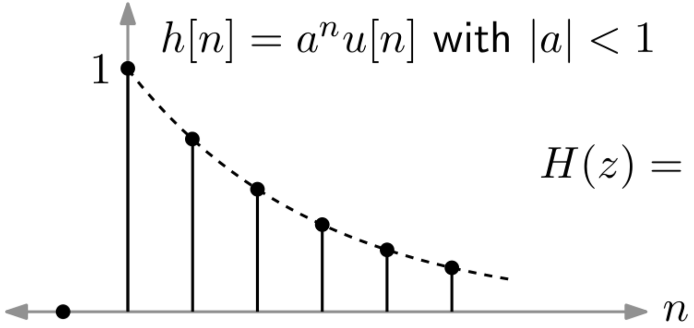
$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

$\mathcal{Z}^{-1}\{\cdot\}$

$$x[n] = 2\,\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8\,u[n]$$

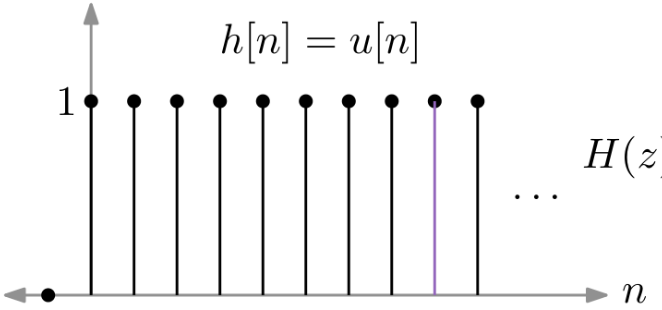
Known z-transform pairs:



$h[n] = a^n u[n]$ with $|a| < 1$

$$H(z) = \frac{1}{1 - az^{-1}}$$

$|z| > |a|$



$h[n] = u[n]$

$$H(z) = \frac{1}{1 - z^{-1}}$$

$|z| > 1$

Partial fraction expansion steps

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = C + Dz^{-2} + \frac{E}{1 - Fz^{-1}} + \frac{Gz^{-1}}{(1 - Kz^{-1})^2} + \ldots$$

1. If $M \geq N$, use long division to get to $M < N$ (do this with powers of z^{-1})
2. Convert equation to have positive powers of z
3. Factorise $X(z)/z$
4. Do partial fraction expansion
5. Convert back to powers of z^{-1}
6. Inverse by inspection using known z-transform pairs

1. Long division

For example, we want to go from

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \quad \left\{ \begin{array}{l} M = 3 \\ N = 2 \\ M \geq N \end{array} \right.$$

to

$$X(z) = 1 + 2z^{-1} + \left[\frac{\frac{1}{6}z^{-1}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \right] \quad M < N$$

Long division recap by Barry Van Veen: https://youtu.be/aelioE_4Wuc&t=707

4. Partial fraction expansion

If poles are distinct, expand like this:

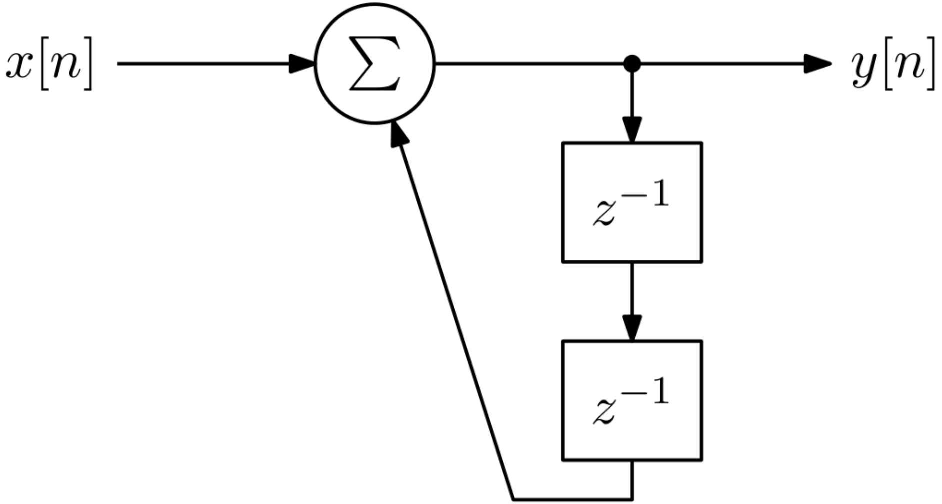
$$\frac{\cdots}{(z-p_1)(z-p_2)\cdots(z-p_N)} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_N}{z-p_N}$$

If there are repeated poles, expand like this:

$$\frac{\cdots}{(z-p_r)^R} = \frac{A_{r_1}}{(z-p_r)} + \frac{A_{r_2}}{(z-p_r)^2} + \cdots + \frac{A_{r_R}}{(z-p_r)^R}$$

$$\frac{\cdots}{(z-1)^3} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z-1)^3}$$

Inverse z-transform example



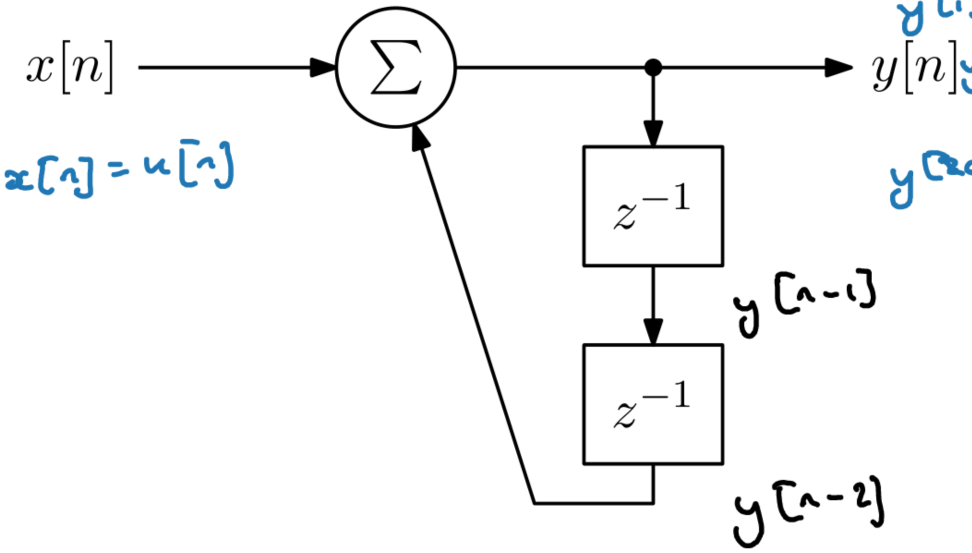
(a) Determine an expression for $H(z)$

(b) Determine an expression for $Y(z)$ when $x[n] = u[n]$

(c) Determine a closed-form expression for $y[n]$ from $Y(z)$

$\mathcal{Z}^{-1}\{\cdot\}$

(a) Determine an expression for $H(z)$:



$x[n] = u[n]$

$y[0] = 1$

$y[1] = 1$

$y[2] = 2$

$\vdots$

$y[200] = ?$

$$Y(z) = X(z) \cdot H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$y[n] = x[n] + y[n-2]$$

$$\downarrow \mathcal{Z}$$

$$Y(z) = X(z) + z^{-2} Y(z)$$

$$Y(z)\,(1 - z^{-2}) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-2}}$$

(b) Determine an expression for $Y(z)$ when $x[n] = u[n]$:

$Y(z) = H(z) \cdot X(z)$

$$H(z) = \frac{1}{1 - z^{-2}}$$

$X(z) = \frac{1}{1 - z^{-1}}$, $|z| > 1$

$$Y(z) = H(z) \cdot X(z)$$

$$= \frac{1}{1 - z^{-2}} \cdot \frac{1}{1 - z^{-1}} \quad , \quad |z| > 1$$

$\mathcal{Z}^{-1}$

$y[n] = ?$

(c) Determine a closed-form expression for $y[n]$ from $Y(z)$:

$$Y(z) = \underbrace{\frac{1}{1-z^{-1}}}_{X(z)} \underbrace{\frac{1}{1-z^{-2}}}_{H(z)} \times \frac{z^3}{z \cdot z^2} = \frac{z^3}{(z-1)(z^2-1)}$$

2

$$= \frac{1}{1-z^{-1}-z^{-2}+z^{-3}}$$

1. Long division to get $M < N$ (do this with powers of z^{-1}) ✓
2. Convert equation to have positive powers of z ✓
3. Factorise $X(z)/z$ ✓

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z^2-1)}$$

$$= \frac{z^2}{(z-1)(z+1)(z-1)}$$

$$= \frac{z^2}{(z+1)(z-1)^2}$$

4. Partial fraction expansion

$$\frac{Y(z)}{z} = \left[\frac{z^2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}\right]$$

$$z^2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1) \quad \ldots \text{①}$$

$z=-1$ in ①: $(-1)^2 = A(-1-1)^2 + B(0)(-2) + C(0)$

$1 = A(-2)^2 \quad \therefore \; A = \frac{1}{4}$

$z=1$ in ①: $(1)^2 = 2C \quad \therefore \; C = \frac{1}{2}$

$z=0$ in ①: $0 = A(-1)^2 + B(1)(-1) + C(1)$

$B = A + C \quad \therefore \; B = \frac{3}{4}$

$$\therefore \; \frac{Y(z)}{z} = \frac{\frac{1}{4}}{z+1} + \frac{\frac{3}{4}}{z-1} + \frac{\frac{1}{2}}{(z-1)^2}$$

$$Y(z) = \frac{\frac{1}{4}z}{z+1} + \frac{\frac{3}{4}z}{z-1} + \frac{\frac{1}{2}z}{(z-1)^2}$$

5. Convert back to powers of z^{-1}

z^2-2z+1

$$Y(z) = \frac{\frac{1}{4}z}{z+1} \times \frac{z^{-1}}{z^{-1}} + \frac{\frac{3}{4}z}{z-1} \times \frac{z^{-1}}{z^{-1}} + \frac{\frac{1}{2}z}{(z-1)^2} \times \frac{z^{-2}}{z^{-2}}$$

$$= \frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{3}{4}}{1-z^{-1}} + \frac{\frac{1}{2}z^{-1}}{1-2z^{-1}+z^{-2}}$$

$$= \frac{\frac{1}{4}\cdot 1}{1+z^{-1}} + \frac{\frac{3}{4}\cdot 1}{1-z^{-1}} + \frac{\frac{1}{2}z^{-1}}{(1-z^{-1})^2}, \quad |z|>1$$

6. Inverse by inspection using known z-transform pairs

$$y[n] = \frac{1}{4}(-1)^n u[n] + \frac{3}{4}u[n] + (1)\frac{1}{2}n\,u[n]$$

$$= \left[\frac{(-1)^n}{4} + \frac{3}{4} + \frac{1}{2}n\right]\cdot u[n]$$

$y[0] = 1$

$y[1] = -\frac{1}{4} + \frac{3}{4} + \frac{1}{2} = 1$

$y[2] = \frac{1}{4} + \frac{3}{4} + 1 = 2$

$\vdots$

$y[200] = ?$

Complex conjugate poles

If $x[n]$ is a real signal, any complex poles in its z-transform $X(z)$ will occur in conjugate pairs:

$$X(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

$$p = r e^{j\omega}$$

$$A = a \cdot e^{j\phi}$$

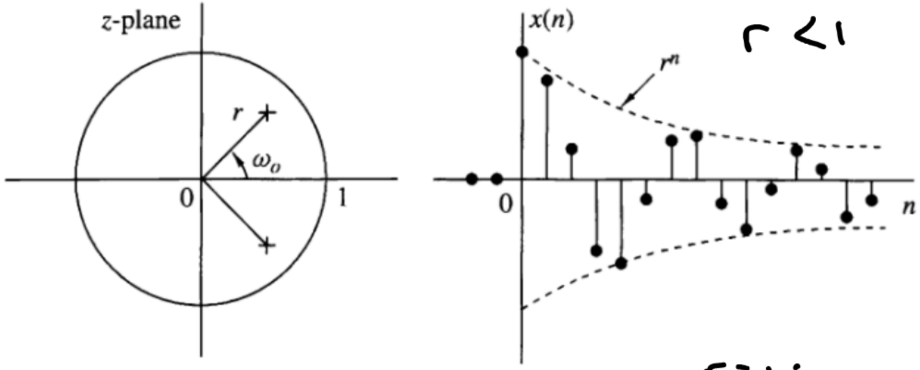
$$\Longleftrightarrow$$

$$x[n] = 2 a r^n \cos(\omega n + \phi) \cdot u[n]$$

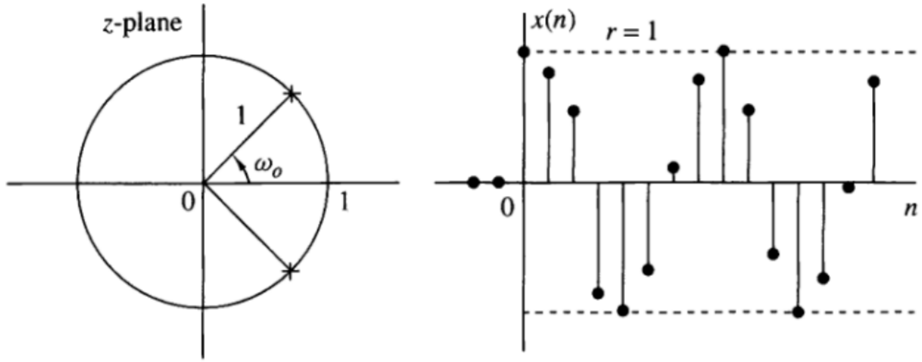
With $p = re^{j\omega}$ and $A = ae^{j\phi}$:

$$2u[n] \cdot ar^n \cos(\omega n + \phi) \qquad \Leftrightarrow \qquad \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

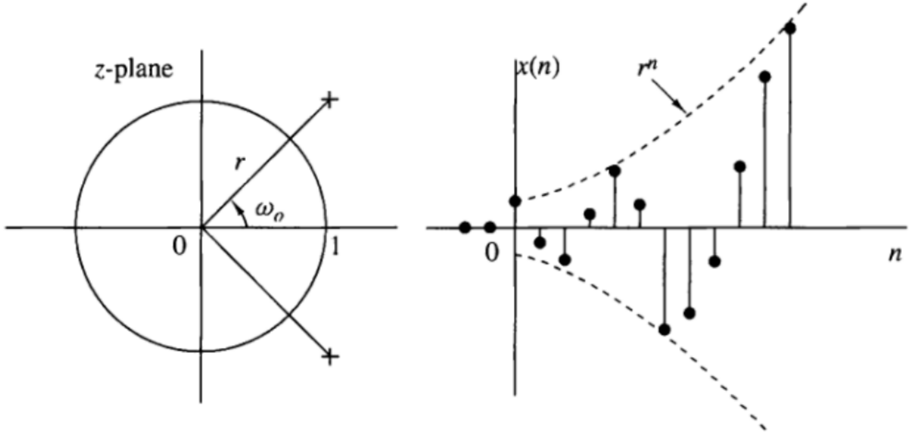
$r < 1$



$r = 1$:



$r > 1$



Further watching and reading

Section 3.4.3 of Proakis and Manolakis (2007)

Barry Van Veen's videos on the z-transform:
https://www.youtube.com/playlist?list=PLGI7M8vwfrFNvNxfQGXntdwQ2lRSe0frf