Introduction to the z-transform

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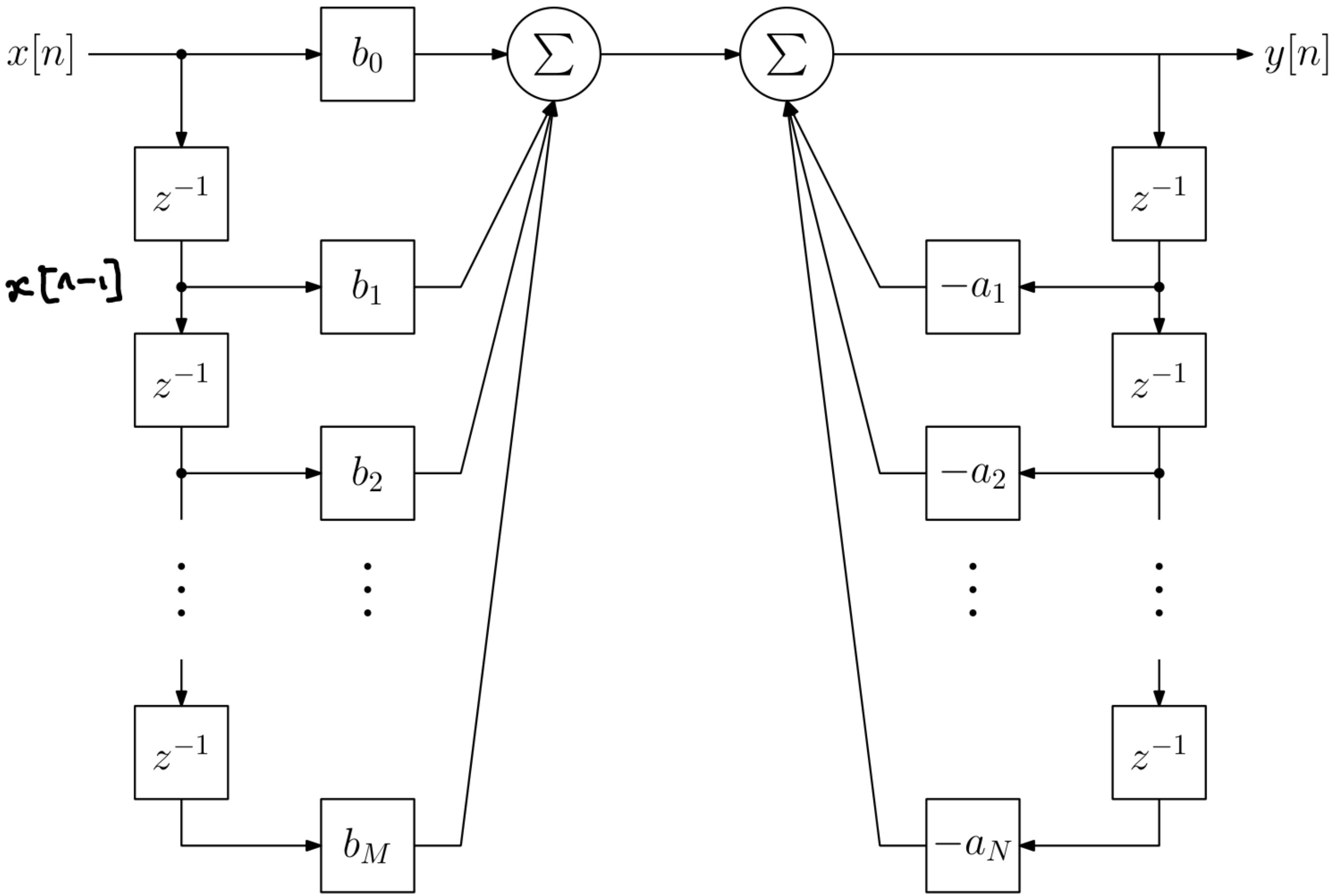
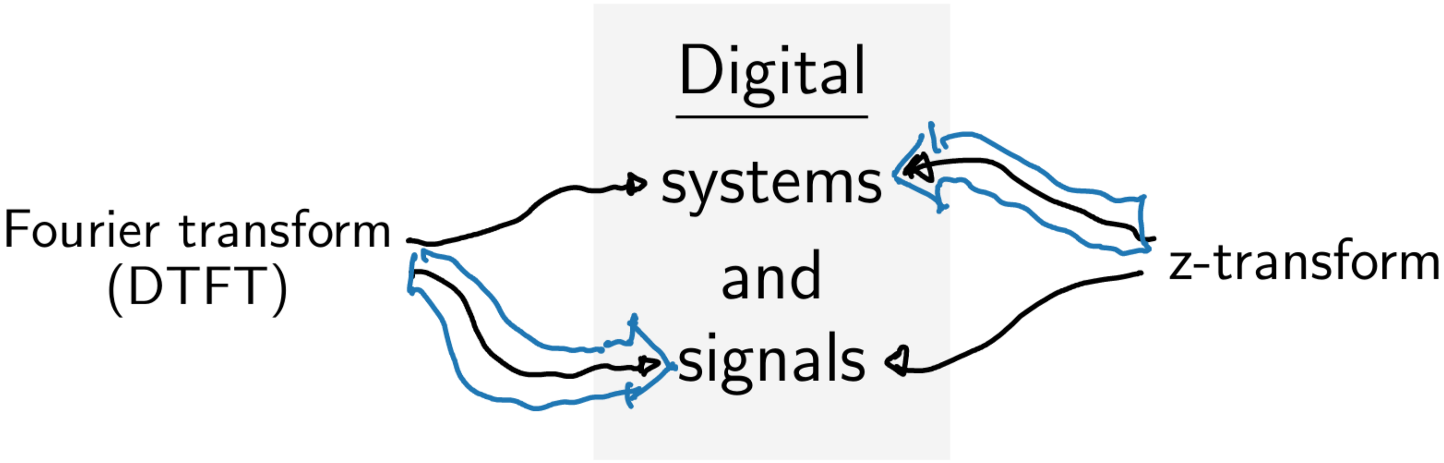
How do we know what a discrete system does to a signal?



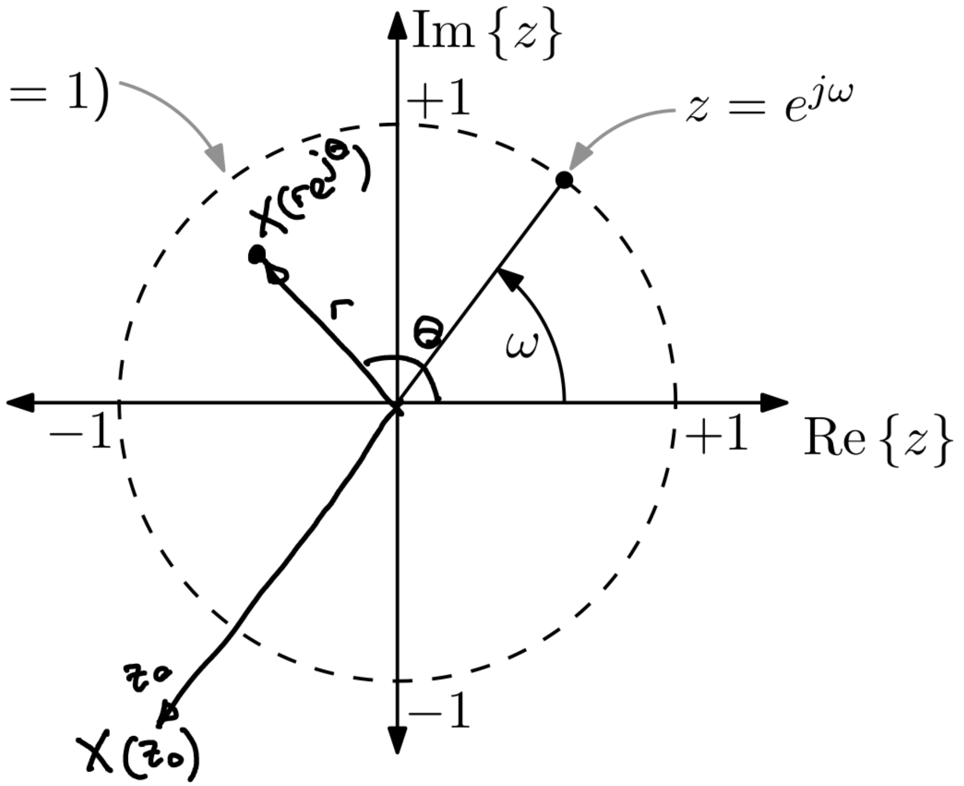

Digital
systems
and
signals
Fourier transform
(DTFT)
z-transform

The z-transform

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

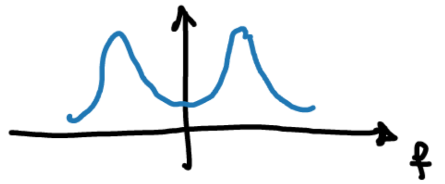
$$X(z) = \mathcal{Z}\{x[n]\}, \quad z \in \mathbb{C}$$

Unit circle
(contour with $|z| = 1$)



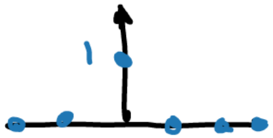
Fourier transform:

$$X(f) = \mathcal{F}\{x(t)\}$$



Simple z-transform examples

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

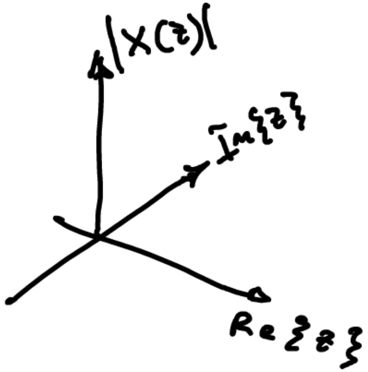


$x[n] = \delta[n] \iff X(z) = \ldots + 0 + 0 + 1 + 0 + 0 = 1$

$x[n] = \{\underset{\uparrow}{\overset{x[0]}{1}}, \overset{x[1]}{2}, \overset{x[2]}{3}, \overset{x[3]}{3}\} \iff X(z) = 1 + 2z^{-1} + 3z^{-2} + 3z^{-3}$

$x[n] = \{\overset{x[-2]}{1}, \overset{x[-1]}{2}, \underset{\uparrow}{\overset{x[0]}{5}}, 7, 0\} \iff X(z) = 1 \cdot z^{-(-2)} + 2z^{+1} + 5 + 7z^{-1} + 0 \ldots.$

A function taking a complex value
and producing a complex value:



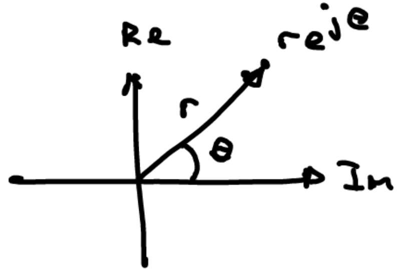
$$X(z) = z$$

$$X(z) = \frac{1}{z - 0.58}$$

Region of convergence (ROC)

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

The z-transform exists only for those values of z for which the infinite sum converges. For a particular signal $x[n]$, the values of z for which this is true is the region of convergence (ROC) of the z-transform $X(z)$.



To find an expression for the ROC, write z in polar form:

$$z = re^{j\theta} \text{ with } \ r = |z| \geq 0 \text{ and } \theta = \angle z$$

Substitute into z-transform definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\theta}\right)^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\theta}\right)^{-n}$$

Inside the ROC we require $|X(z)| < \infty$:

$$\begin{aligned}
|X(z)| &= \left| \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\theta n} \right| \\
&\leq \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} e^{-j\theta n} \right| \\
&= \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| \\
&= \sum_{n=-\infty}^{-1} \left| x[n] r^{-n} \right| + \sum_{n=0}^{\infty} \left| x[n] r^{-n} \right| \\
&= \sum_{n_1=1}^{\infty} \left| x[-n_1] r^{n_1} \right| + \sum_{n=0}^{\infty} \left| x[n] r^{-n} \right|
\end{aligned}$$

$|a+b+c| \leq |a| + |b| + |c|$

$|e^{-j\theta n}| = 1$

$n_1 = -n$

$n = -n_1$

Causal, anti-causal and general signals

$$|X(z)| \leq \sum_{n=1}^{\infty} |x[-n] r^{n}| + \sum_{n=0}^{\infty} |x[n] r^{-n}|$$

Causal:

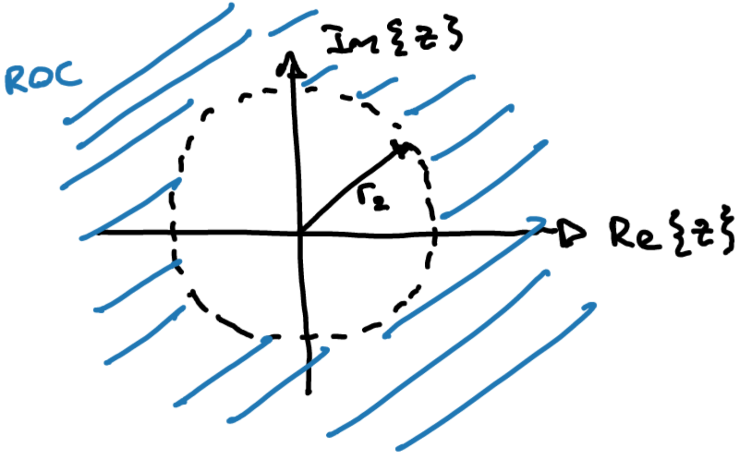
$$|X(z)| \leq \sum_{n=0}^{\infty} |x[n] \cdot r^{-n}|$$

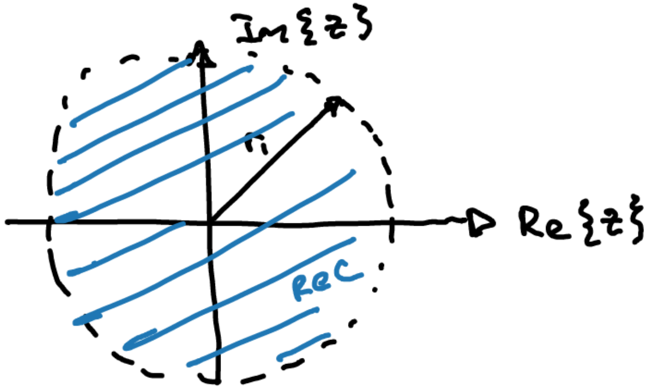
$$= \sum_{n=0}^{\infty} \left| x[n] \cdot \frac{1}{r^n} \right|$$

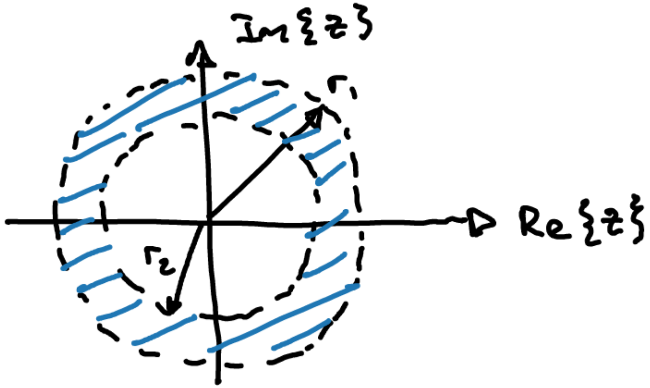
Anti-causal:

$$|X(z)| \leq \sum_{n=1}^{\infty} |x[-n] \cdot r^{n}|$$

General:



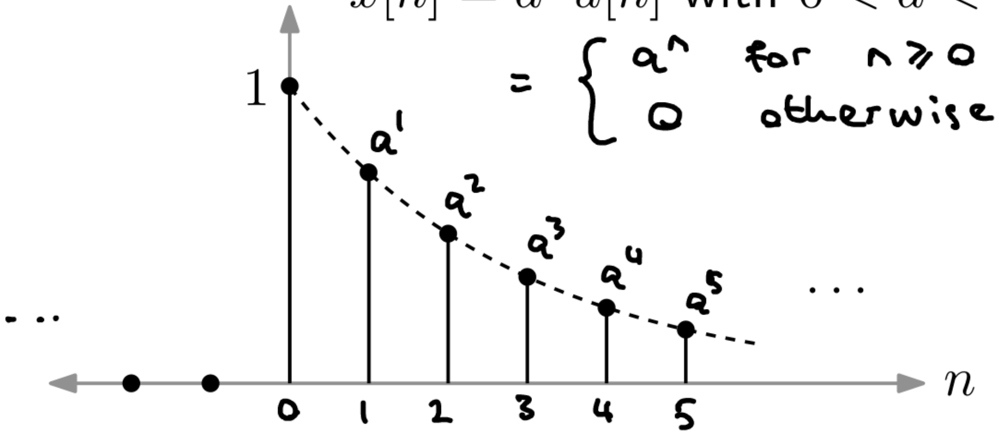




Another z-transform example

$x[n] = a^n u[n]$ with $0 < a < 1$

$$= \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Identities:

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1 - b} \quad \text{for } |b| < 1$$

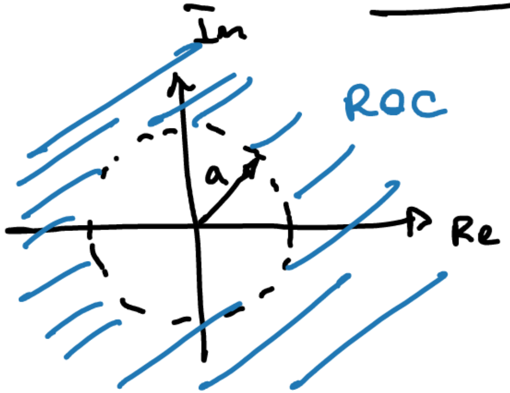
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

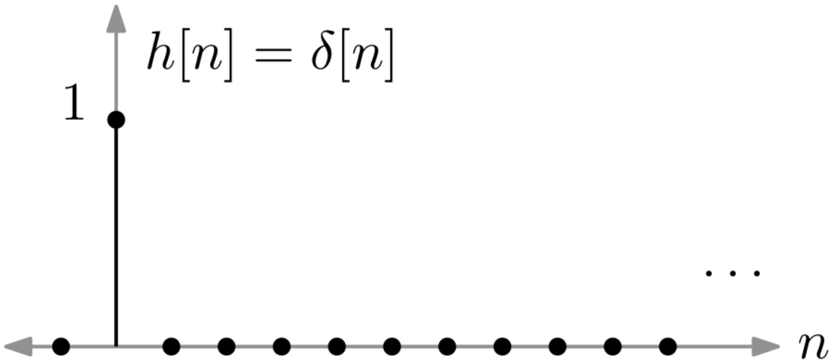
$$= \sum_{n=0}^{\infty} (\underbrace{a z^{-1}}_{b})^n$$

$$= \frac{1}{1 - a z^{-1}}, \quad |a z^{-1}| < 1$$

$$\longrightarrow \quad |a| < |z|$$

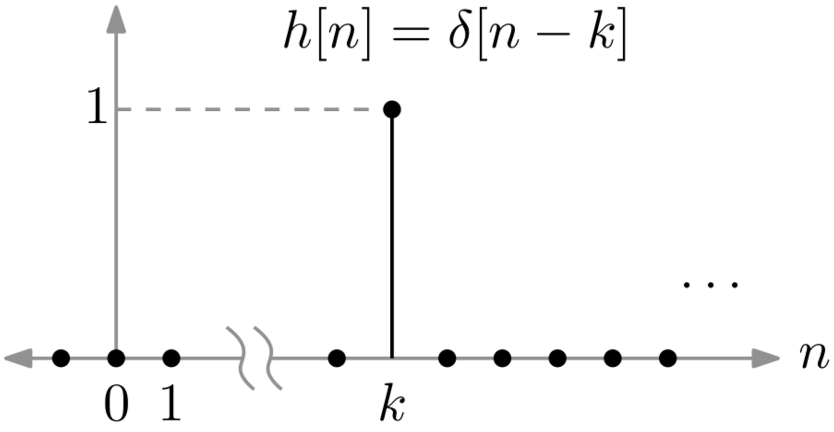


Discrete time-domain $\Leftrightarrow$ z-transform



$$H(z) = 1$$

ROC:
all z

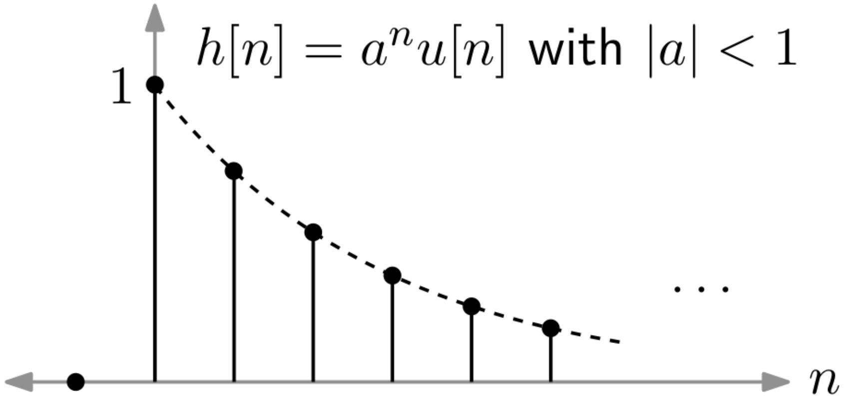


$$H(z) = z^{-k} = \frac{1}{z^k}$$

$z \neq 0$, $k > 0$

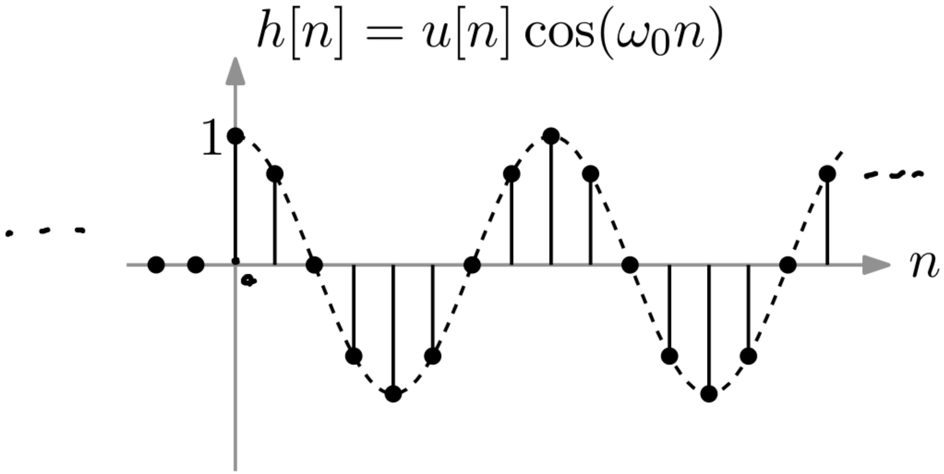
$z \neq \infty$, $k < 0$

Discrete time-domain ⇔ z-transform



$$H(z) = \frac{1}{1 - az^{-1}}$$

$$|z| > |a|$$



$$H(z) = \frac{1 - (\cos\omega_0)z^{-1}}{1 - (2\cos\omega_0)z^{-1} + z^{-2}}$$

$$|z| > 1$$

Properties of the z-transform

- Linearity:

$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha\mathcal{Z}\{x[n]\} + \beta\mathcal{Z}\{y[n]\}$$

- Time shift:

$$\mathcal{Z}\{x[n-k]\} = z^{-k}\mathcal{Z}\{x[n]\}$$

- Convolution:

$$\mathcal{Z}\{x[n] * y[n]\} = \mathcal{Z}\{x[n]\} \cdot \mathcal{Z}\{y[n]\}$$

- Initial-value theorem:

$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{z\to\infty} X(z) = x[0]$$

- Final-value theorem:

$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$$

Final-value theorem example

Theorem:

$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)X(z)$$

Example:

$$x[n] = u[n] \qquad \Leftrightarrow \qquad X(z) = \frac{1}{1-z^{-1}}$$

$$\lim_{n\to\infty} x[n] = 1$$

$$\lim_{z\to 1} (z-1)\, X(z)$$

$$= \lim_{z\to 1} (z-1) \cdot \frac{1}{1-z^{-1}}$$

$$= \lim_{z\to 1} \frac{z\,\cancel{(1-z^{-1})}}{\cancel{(1-z^{-1})}}$$

$$= \lim_{z\to 1} z = 1$$