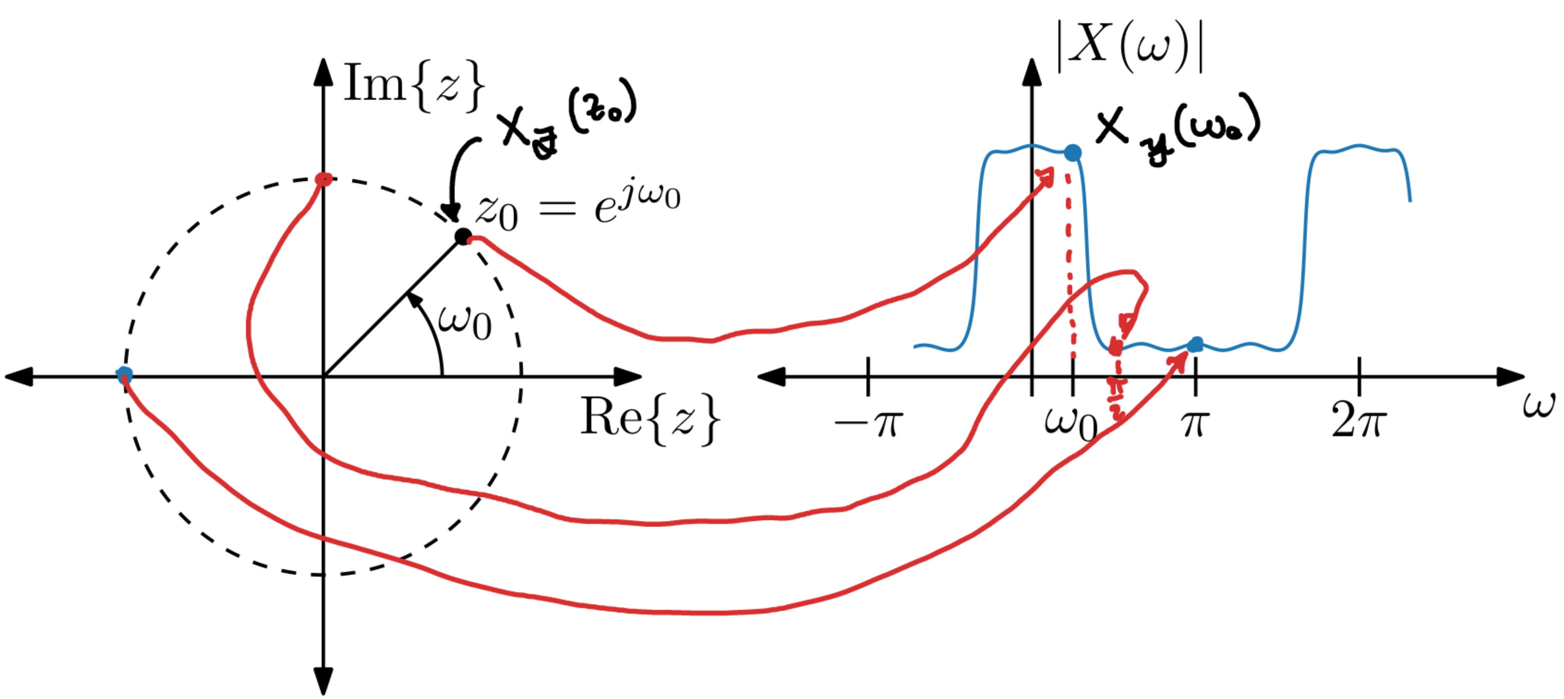
Frequency response with the z-transform

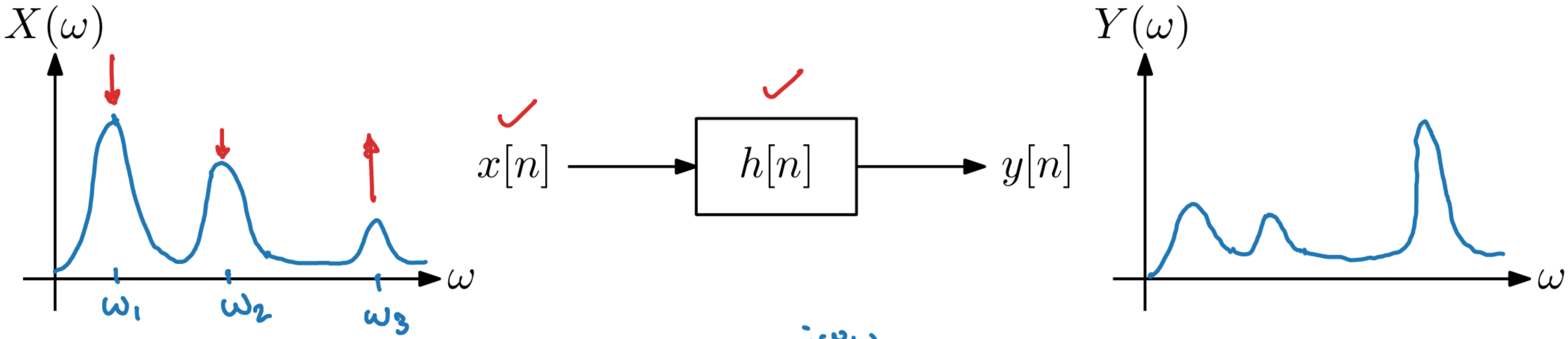
Herman Kamper

Relationship between z-transform and Fourier transform

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$X_{\mathcal{F}}(\omega) = X_{\mathcal{Z}}(z)\Big|_{z=e^{j\omega}} = X_{\mathcal{Z}}(e^{j\omega})$$

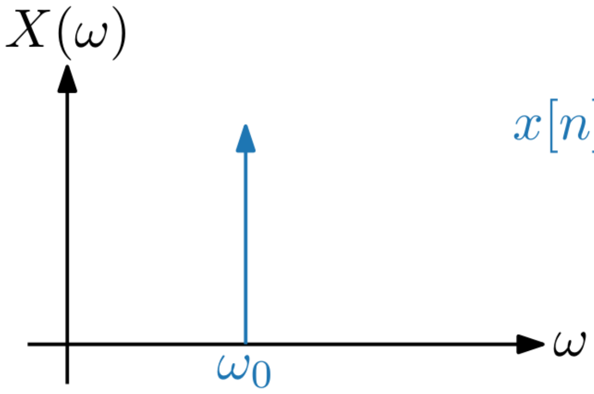




$H(e^{j\omega_1})$

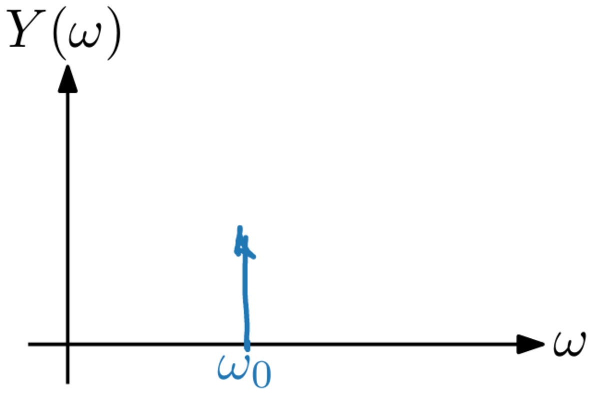
E.g. guitar chord

Frequency response



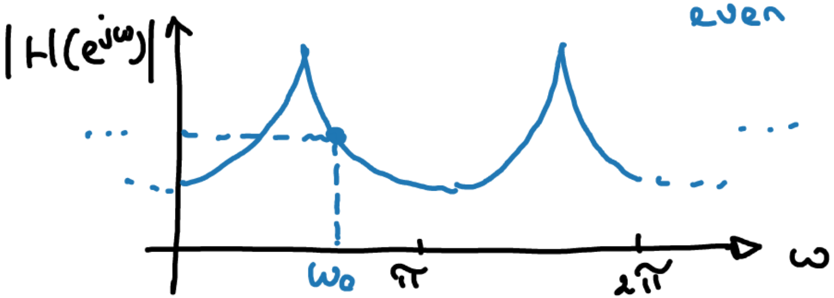
$x[n] = e^{j\omega_0 n}$

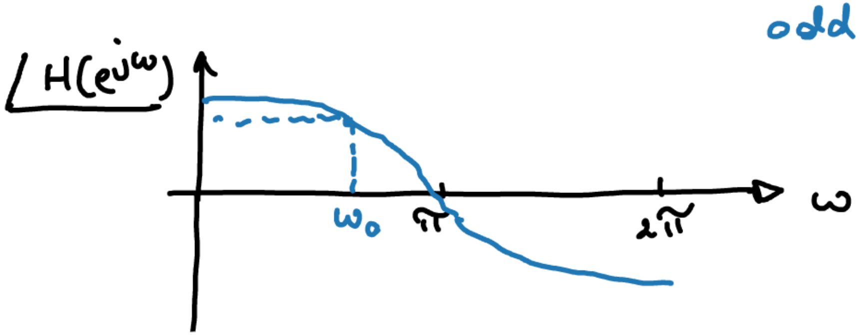
$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$



$$y[n] = H_{\mathcal{F}}(\omega_0)e^{j\omega_0 n}$$

$$= H_{\mathcal{F}}(e^{j\omega_0}) \cdot e^{j\omega_0 n} = |H(e^{j\omega_0})| \cdot e^{j\omega_0 n + \angle H(e^{j\omega_0})}$$





$$x[n] = e^{j\omega n}$$

$$y[n] = x[n] * h[n] \quad = \sum_{i=-\infty}^{\infty} h[i] \cdot x[n-i]$$

$$= \sum_{i=-\infty}^{\infty} h[i]\, e^{j\omega(n-i)}$$

$$= \left[\sum_{i=-\infty}^{\infty} h[i] e^{-j\omega i} \right] \cdot e^{j\omega n}$$

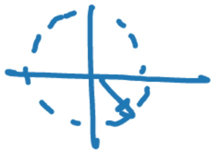
$$= H_y(\omega) \cdot e^{j\omega n}$$

$$= H_{\mathcal{Z}}(e^{j\omega}) \cdot e^{j\omega n}$$

Frequency response: z-plane interpretation

LCCDE:

$$H(z) = b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$



$|e^{j\omega k}| = 1$

Frequency response:

$$H(e^{j\omega}) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

Magnitude response:

$$\left|H(e^{j\omega})\right| = |b_0| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - z_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

Phase response:

$$\angle H(e^{j\omega}) = \angle b_0 + \omega(N - M) + \sum_{k=1}^{M} \angle(e^{j\omega} - z_k) - \sum_{k=1}^{N} \angle(e^{j\omega} - p_k)$$

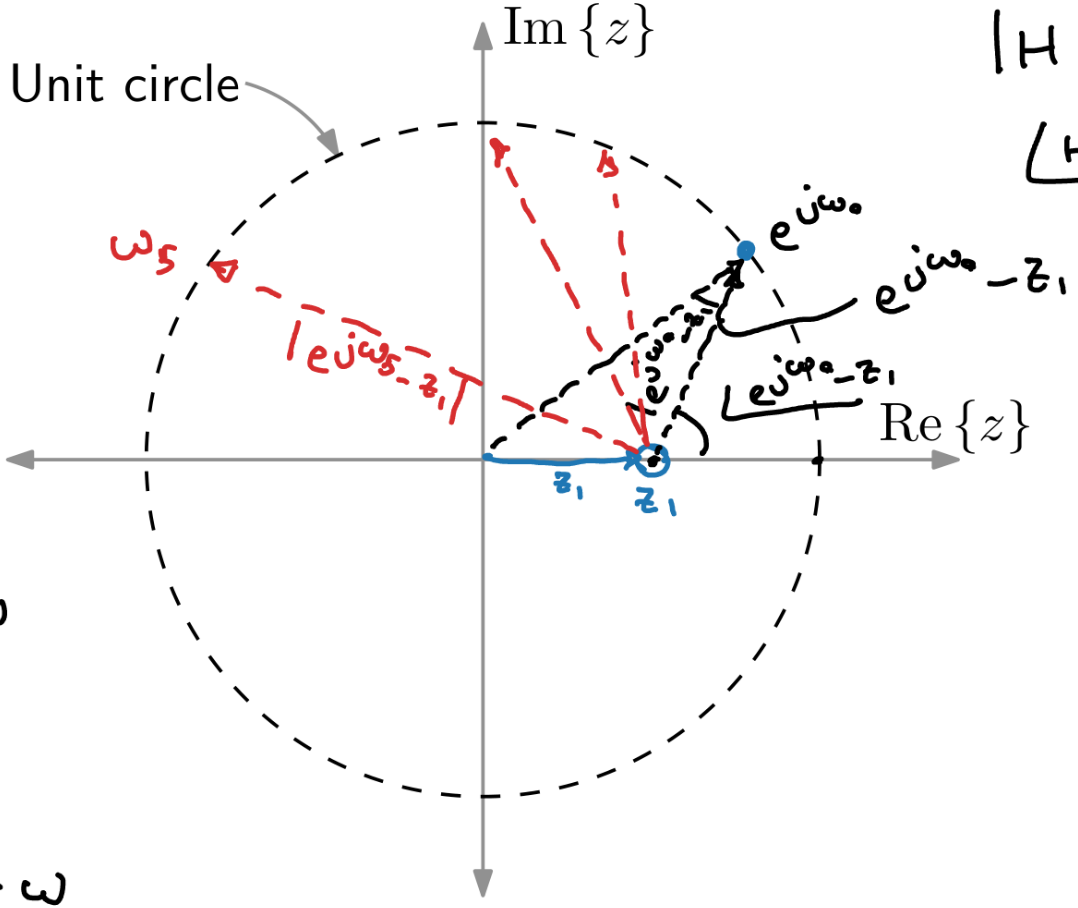
Frequency response examples

$$H(z) = z - z_1$$

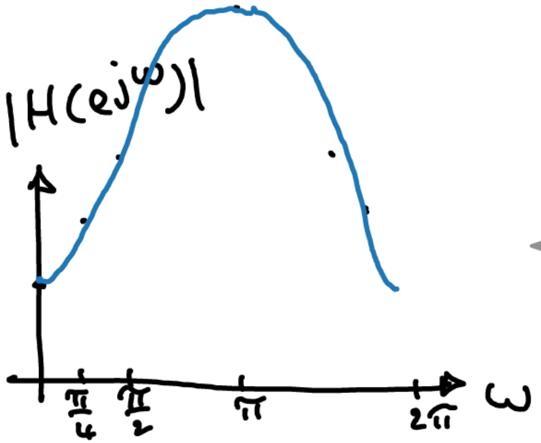
$$H(e^{j\omega}) = e^{j\omega} - z_1$$

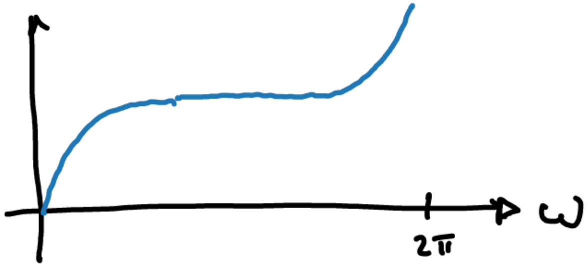
$$|H(e^{j\omega})| = |e^{j\omega} - z_1|$$

$$\angle H(e^{j\omega}) = \angle e^{j\omega} - z_1$$

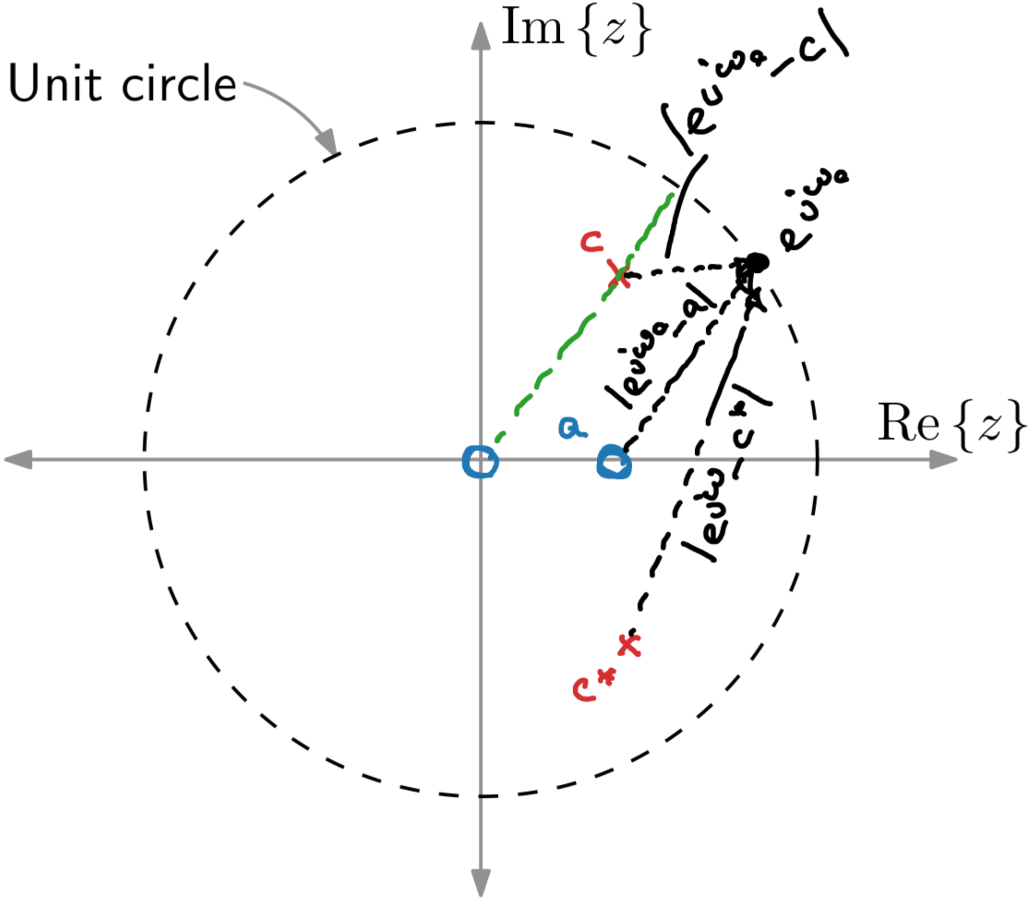


$$e^{j\omega_0} - z_1$$



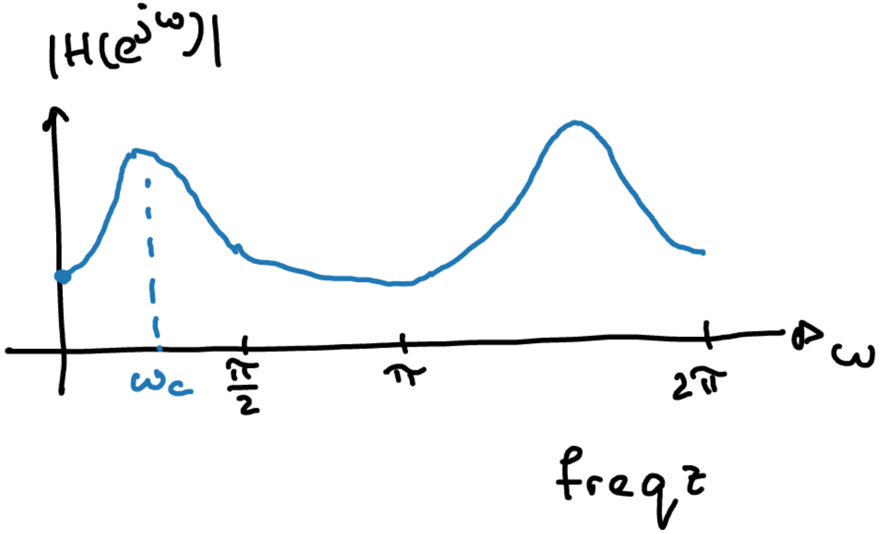


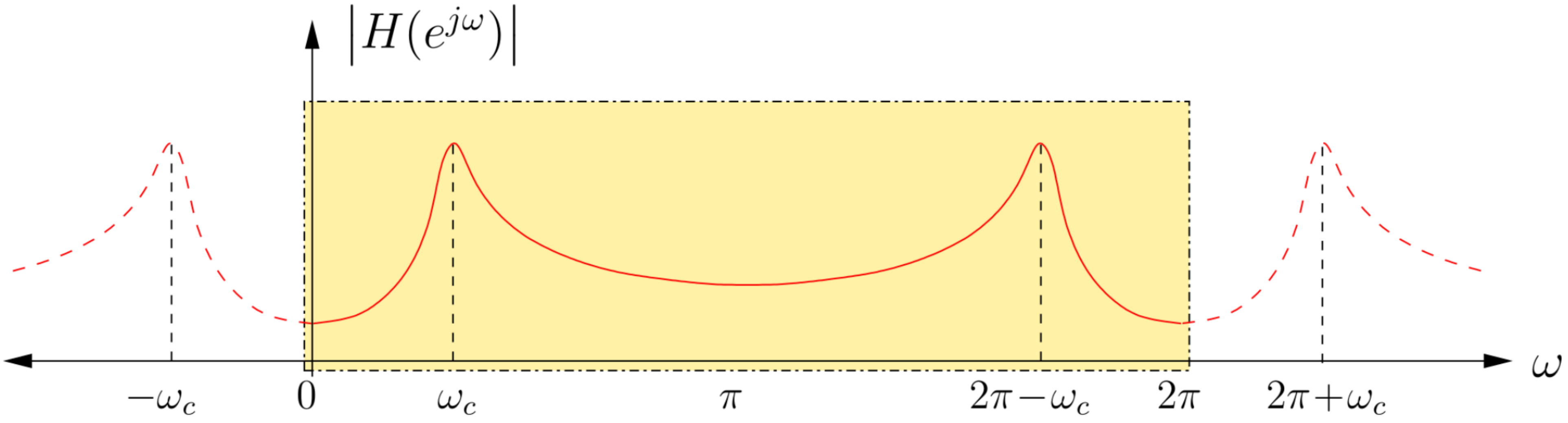
$$H(z) = \frac{1 - az^{-1}}{(1 - cz^{-1})(1 - c^* z^{-1})} = \frac{z(z-a)}{(z-c)(z-c^*)}$$



$$H(e^{j\omega}) = \frac{e^{j\omega}(e^{j\omega} - a)}{(e^{j\omega} - c)(e^{j\omega} - c^*)}$$

$$|H(e^{j\omega})| = \frac{|e^{j\omega} - a|}{|e^{j\omega} - c|\,|e^{j\omega} - c^*|}$$




$|H(e^{j\omega})|$
ω
$-\omega_c$
0
ω_c
π
$2\pi-\omega_c$
2π
$2\pi+\omega_c$