Linear time-invariant systems

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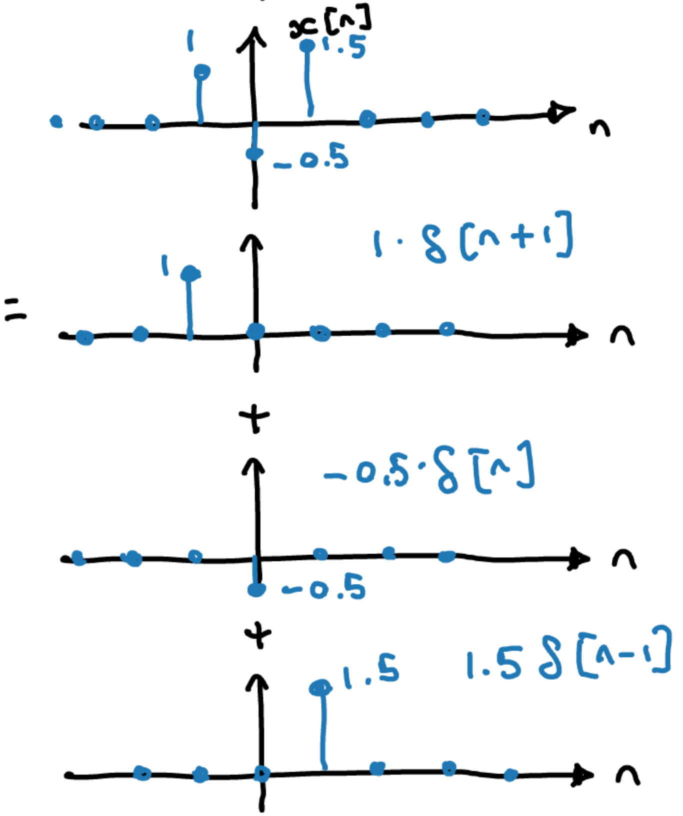
Linear time-invariant (LTI) systems

$$x[n] = \sum_{i=-\infty}^{\infty} x[i] \cdot \delta[n-i]$$



Discrete system: $y[n] = \mathcal{T}\{x[n]\}$

$$= \mathcal{T}\left\{ \sum_{i=-\infty}^{\infty} x[i] \cdot \delta[n-i] \right\}$$

Assume linear: $y[n] = \sum_{i=-\infty}^{\infty} x[i]\, \mathcal{T}\{\delta[n-i]\}$

Assume time-invariant: $h[n] = \mathcal{T}\{\delta[n]\}$

$$\mathcal{T}\{\delta[n-i]\} = h[n-i]$$

LTI: $y[n] = \sum_{i=-\infty}^{\infty} x[i] \cdot h[n-i] = x[n] * h[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

LTI example

$$h[n] = \{\underset{\uparrow}{2} \quad 3 \quad 1 \quad 2\}$$

$$x[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3 \quad 1\}$$

What is $y[n]$?

$$h[i] = \{\underset{\uparrow}{2} \quad 3 \quad 1 \quad 2\}$$

$n=0$: $x[-i] = \{1 \quad 3 \quad 2 \quad \underset{\uparrow}{1}\}$

$n=1$: $x[1-i] = \{1 \quad 3 \quad 2 \quad 1\}$

$$y[n] = x[n] * h[n]$$

$$= \{\underset{\uparrow}{2} \quad 7 \quad 13 \quad 15 \quad 10 \quad 7 \quad 2\}$$

$$x[n] \longrightarrow \overset{?}{\boxed{h[n]}} \longrightarrow y[n]$$

Causality in LTI systems

$$y[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

$$y[\overset{10}{n_0}] = \sum_{i=-\infty}^{\infty} h[i]\cdot x[\overset{10}{n_0} - i]$$

$x[10]$ ✓

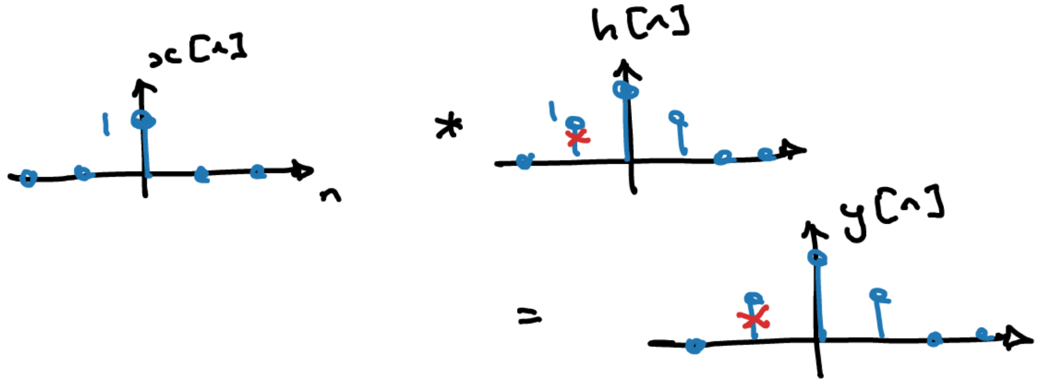
$x[11]$ ✗

Causal system: $y[n_0]$ only depends on $x[n_0 - i]$ where $i \geq 0$

$$y[n_0] = \overbrace{\ldots + h[-2]x[n_0+2] + h[-1]\cdot x[n_0+1]}^{\times} \underbrace{+ h[0]\cdot x[n_0] + h[1]\cdot x[n_0-1] + \ldots}_{\checkmark}$$

$$= \underbrace{\sum_{i=-\infty}^{-1} h[i]\cdot x[n_0-i]}_{=0} + \sum_{i=0}^{\infty} h[i]\cdot x[n_0-i]$$

$\therefore\ h[n] = 0$ for $n < 0$



Causal LTI system:

$$\boxed{h[i] = 0 \text{ for all } i < 0}$$

If we also have that $x[n] = 0$ for $n < 0$, then:

$$\begin{aligned} y[n] &= \sum_{i=0}^{n} h[i]x[n-i] \\ &= \sum_{i=0}^{n} x[i]h[n-i] \end{aligned}$$

BIBO

Stability of LTI systems

$$y[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

$$|y[n]| = \left|\sum_{i=-\infty}^{\infty} h[i]\,x[n-i]\right| = \left|\ \dots + h[-2]\cdot x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \dots\ \right|$$

$$\leq \dots + |h[-2]|\,|x[n+2]| + |h[-1]|\,|x[n+1]| + |h[0]|\,|x[n]| + \dots$$

$$= \sum_{i=-\infty}^{\infty} |h[i]|\cdot|x[n-i]|$$

$|x[n]| \leq M_x$ for all n

$$\leq \sum_{i=-\infty}^{\infty} |h[i]|\cdot M_x$$

$$= M_x \sum_{i=-\infty}^{\infty} |h[i]|$$

We must have $\sum_{i=-\infty}^{\infty} |h[i]| < \infty$

See the note "Necessity, sufficiency and stability"

An LTI system is stable if its impulse response is summable:

$$\boxed{\sum_{i=-\infty}^{\infty} |h[i]| < \infty}$$

From this result it can be shown that:

- $|h[n]| \to 0$ as $n \to \infty$
- $|y[n]| \to 0$ as $n \to \infty$ for finite-duration $x[n]$

Subclasses of LTI systems

Discrete system

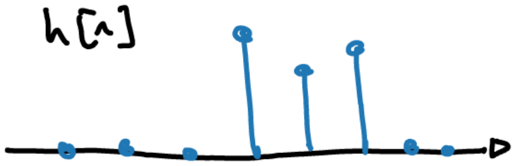
$\downarrow$

LTI system

$\downarrow$

Types of LTI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad y[n] = x[n] * h[n]$$



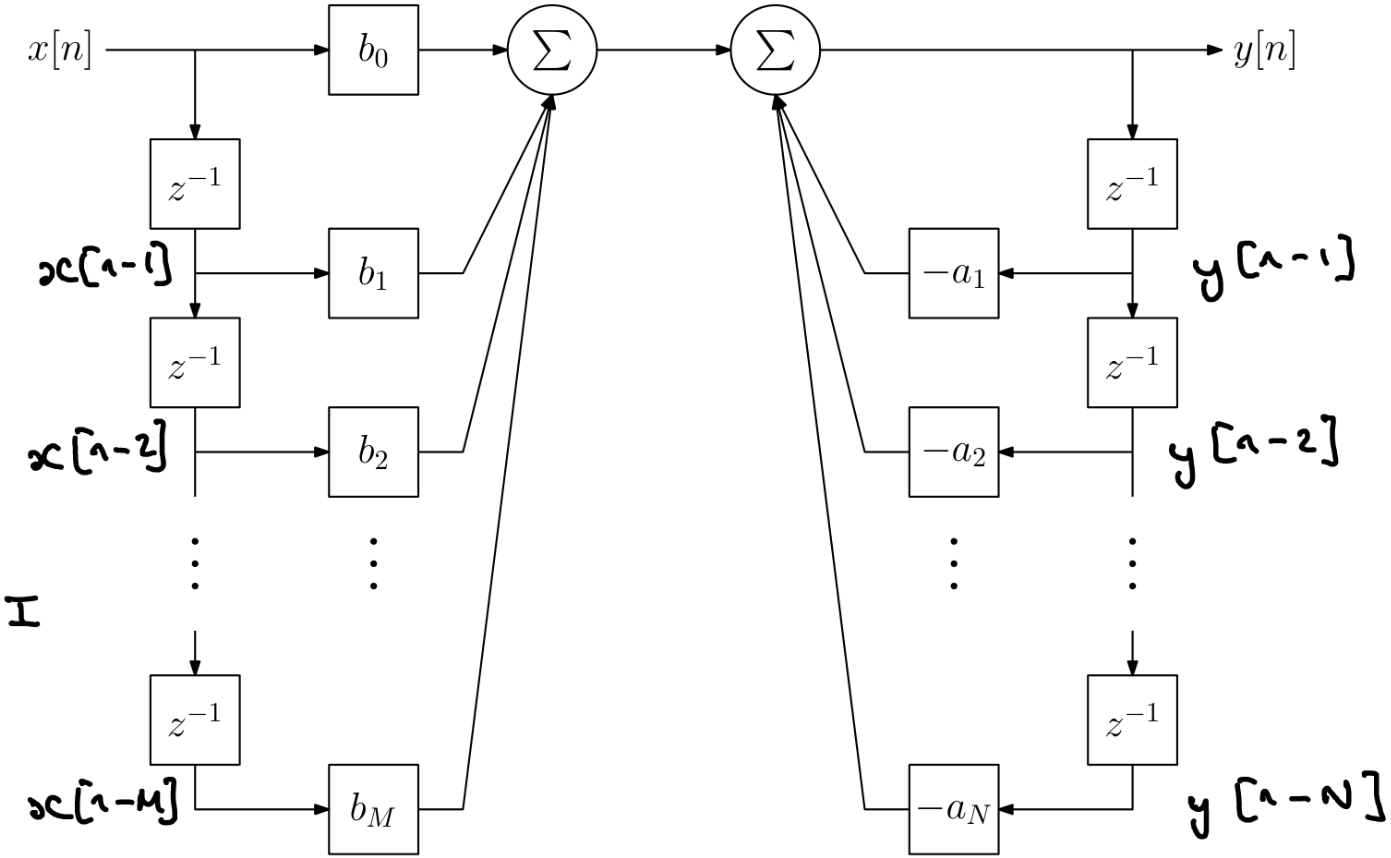
- Finite impulse response (FIR)
  (Moving average)
- Infinite impulse response (IIR) Decaying exponential

- Linear constant-coefficient difference equation (LCCDE): Output is linear combination of finite number of weighted past outputs and past and present inputs

Linear constant-coefficient difference equation (LCCDE)

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

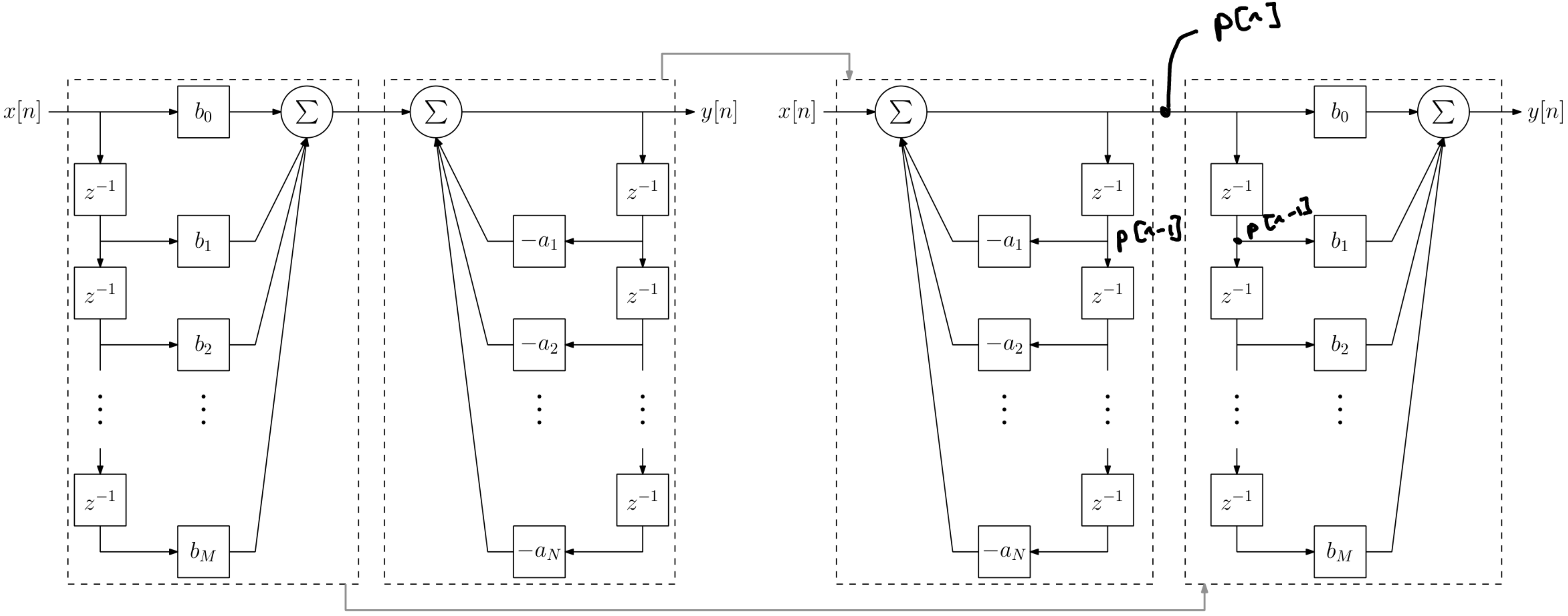
Moving average



Autoregressive

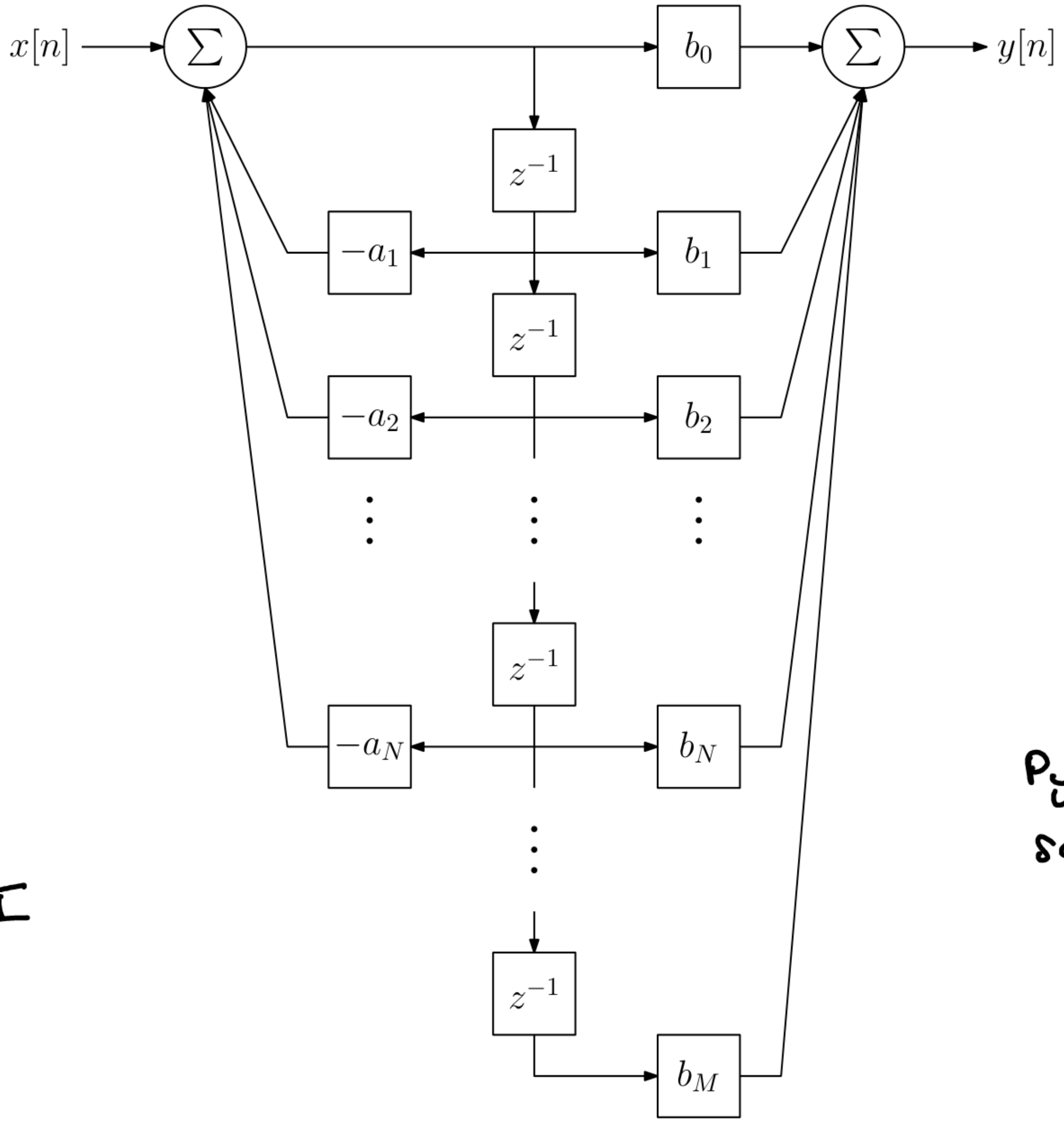
DIRECT FORM I

Efficient LCCDE implementation



$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n] \equiv x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$

$= x[n] * h_1[n] * h_2[n]$



Python:

scipy.signal.lfilter

DIRECT FORM II

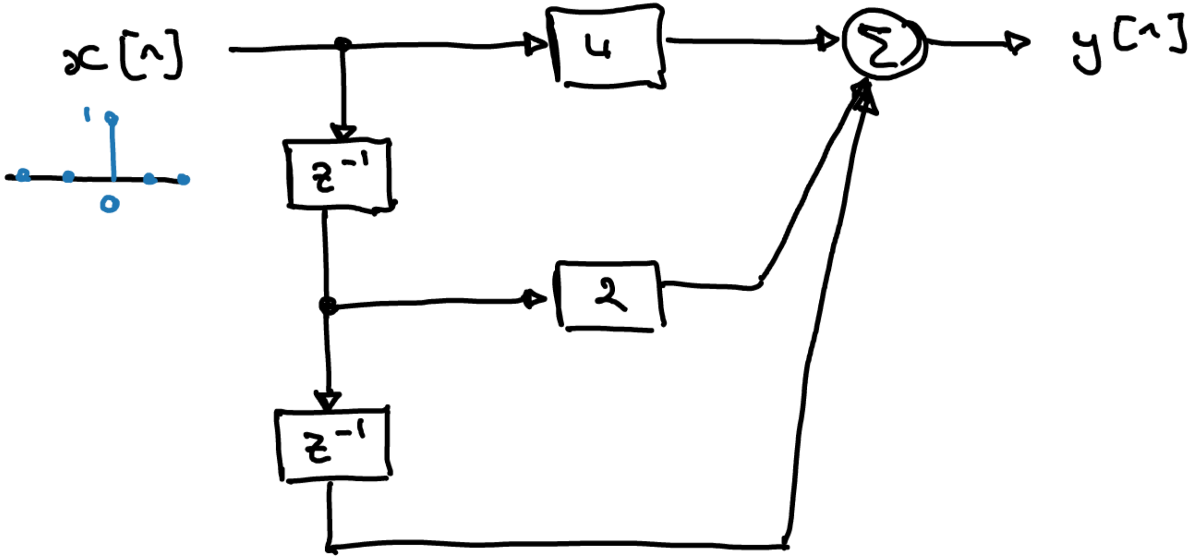
LCCDE example

$$y[n] = 4x[n] + 2x[n-1] + x[n-2]$$

(a) Draw the direct-form I for this filter

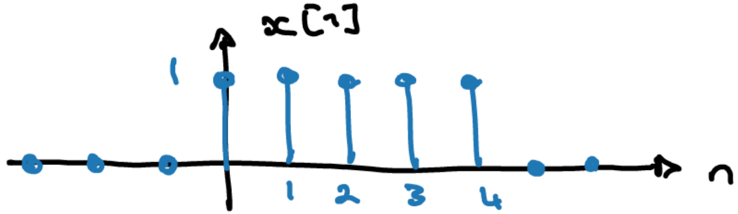
(b) What is the impulse response of this filter?

$$h[n] = \{ \underset{\uparrow}{4} \quad 2 \quad 1 \}$$



$$y[n] = 4x[n] + 2x[n-1] + x[n-2]$$

(c) What is the filter's output for $x[n] = u[n] - u[n-5]$?



$$y[n] = \{ \underset{\uparrow}{4} \quad 6 \quad 7 \quad 7 \quad 7 \quad 3 \quad 1 \}$$

Can do from block diagram

or from $y[n] = x[n] * h[n]$