Transforming between filter types

Simple LPF to HPF transform and more general frequency transforms

Herman Kamper

Discrete LPF

Discrete HPF

BPF

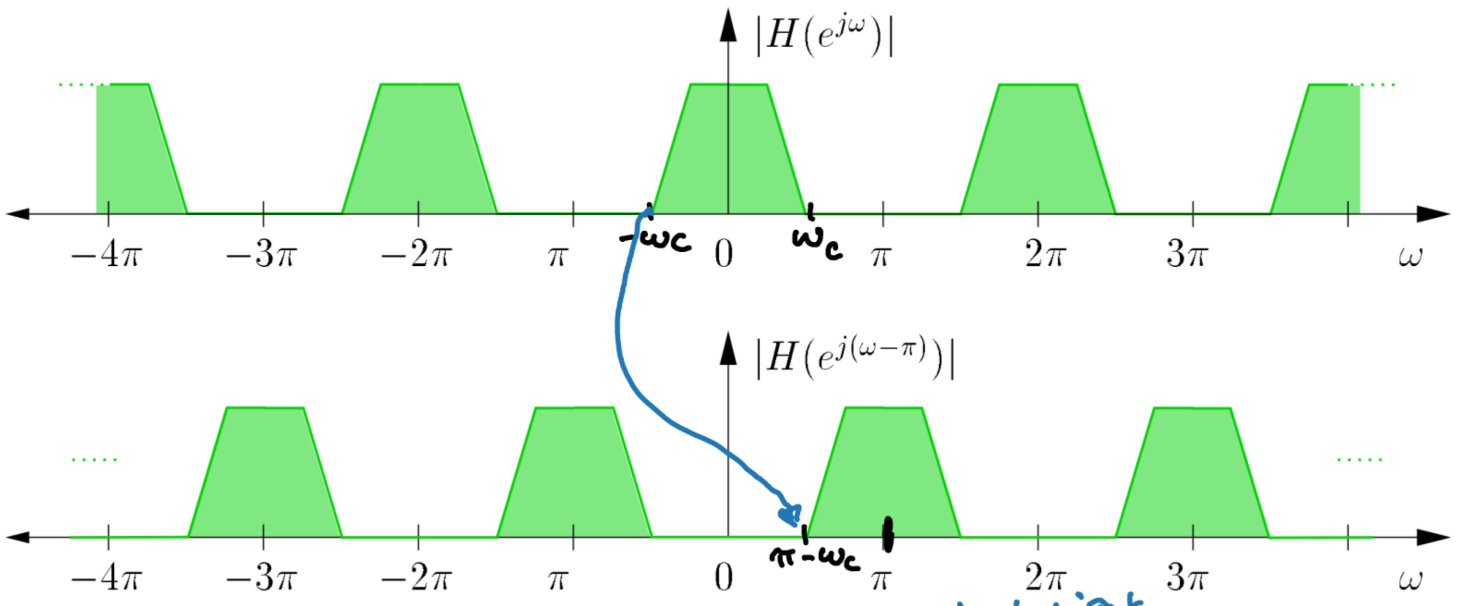
BSF

LPF

Simple low-pass to high-pass transform

$f(t)$

$f(t-a)$



$e^{-j\omega k + j\pi k}$

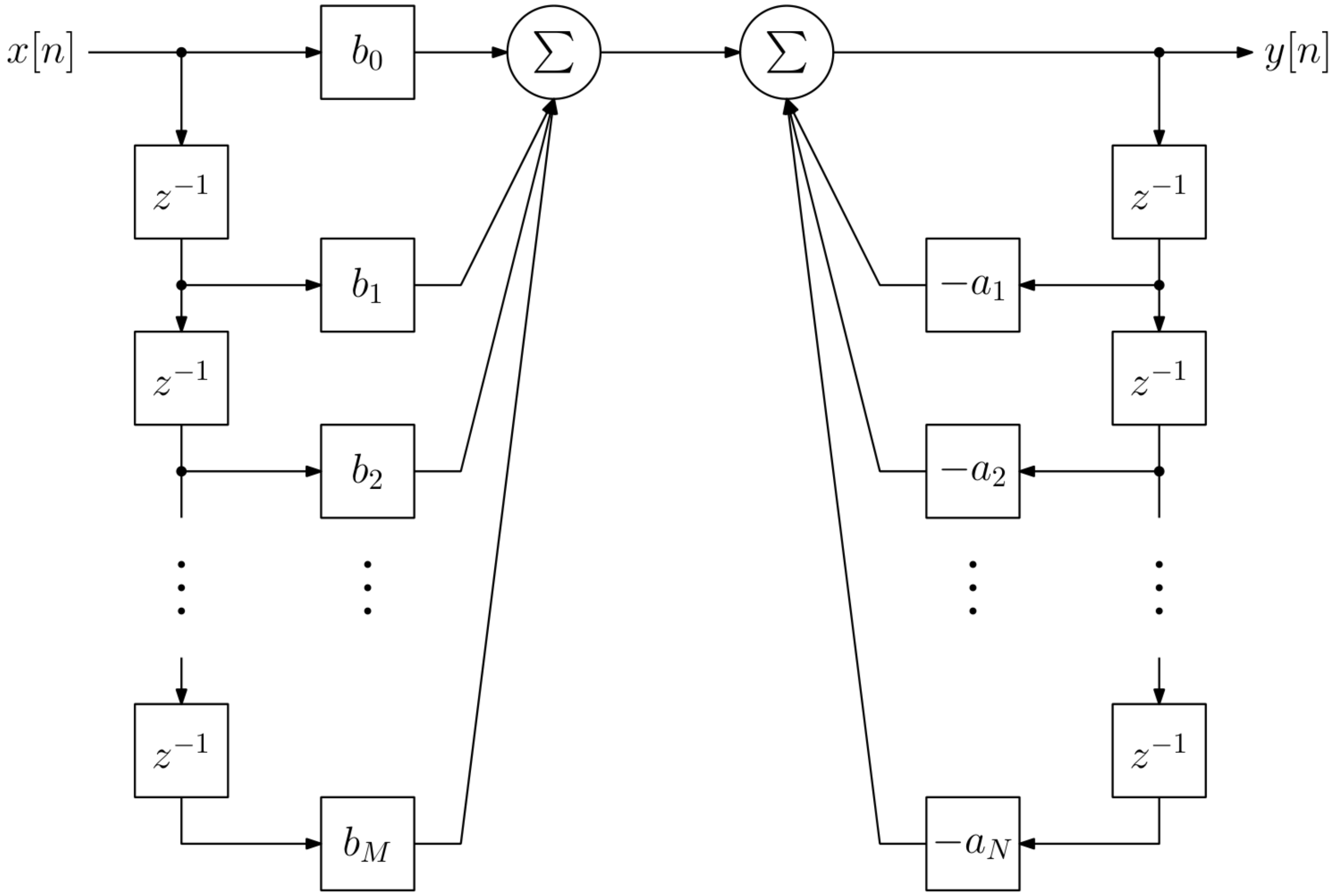
$= e^{-j\omega k} \cdot e^{+j\pi k}$

$= e^{-j\omega k} \cdot (e^{+j\pi})^k = e^{-j\omega k}(-1)^k$

Prototype filter: $H(e^{j\omega}) = \dfrac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$

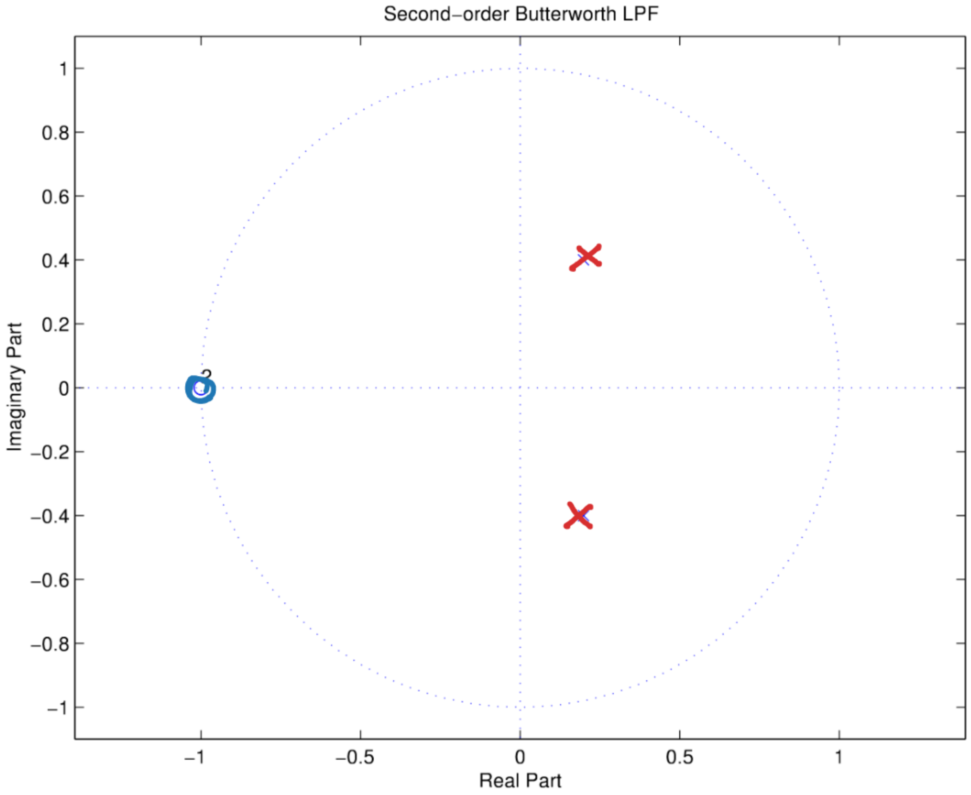
Transformed filter: $H(e^{j(\omega-\pi)}) = \dfrac{\sum_{k=0}^{M} b_k e^{-j(\omega-\pi)k}}{1 + \sum_{k=1}^{N} a_k e^{-j(\omega-\pi)k}} = \dfrac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}}$

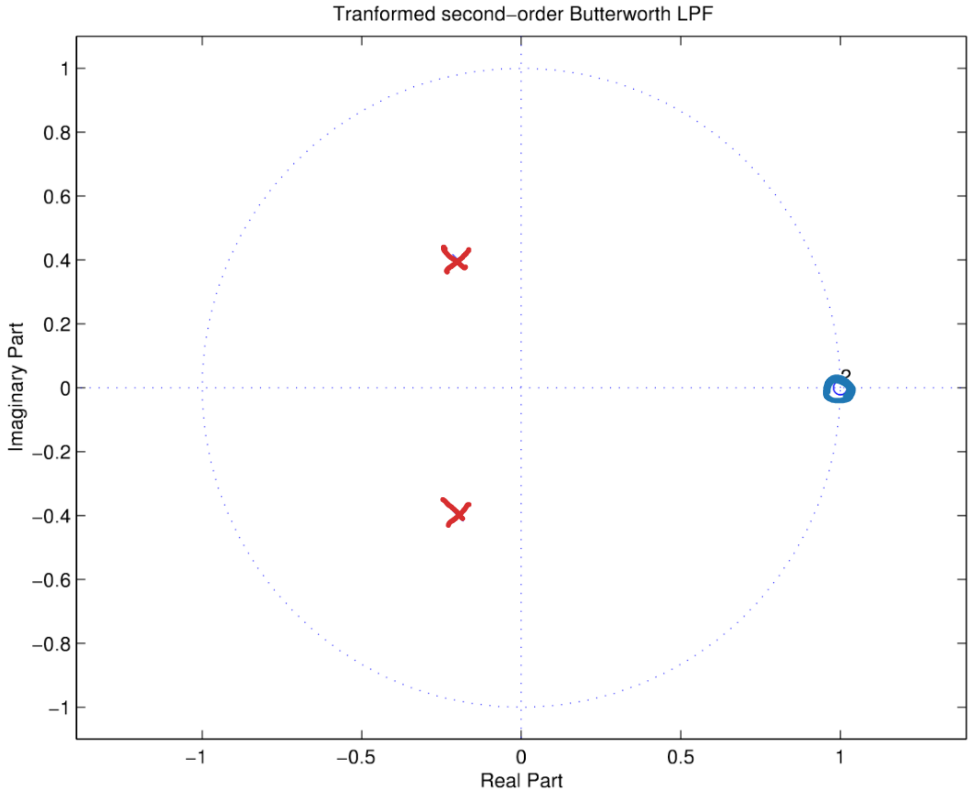
Difference equation: $y[n] = -\sum_{k=1}^{N}(-1)^k a_k y[n-k] + \sum_{k=0}^{M}(-1)^k b_k x[n-k]$

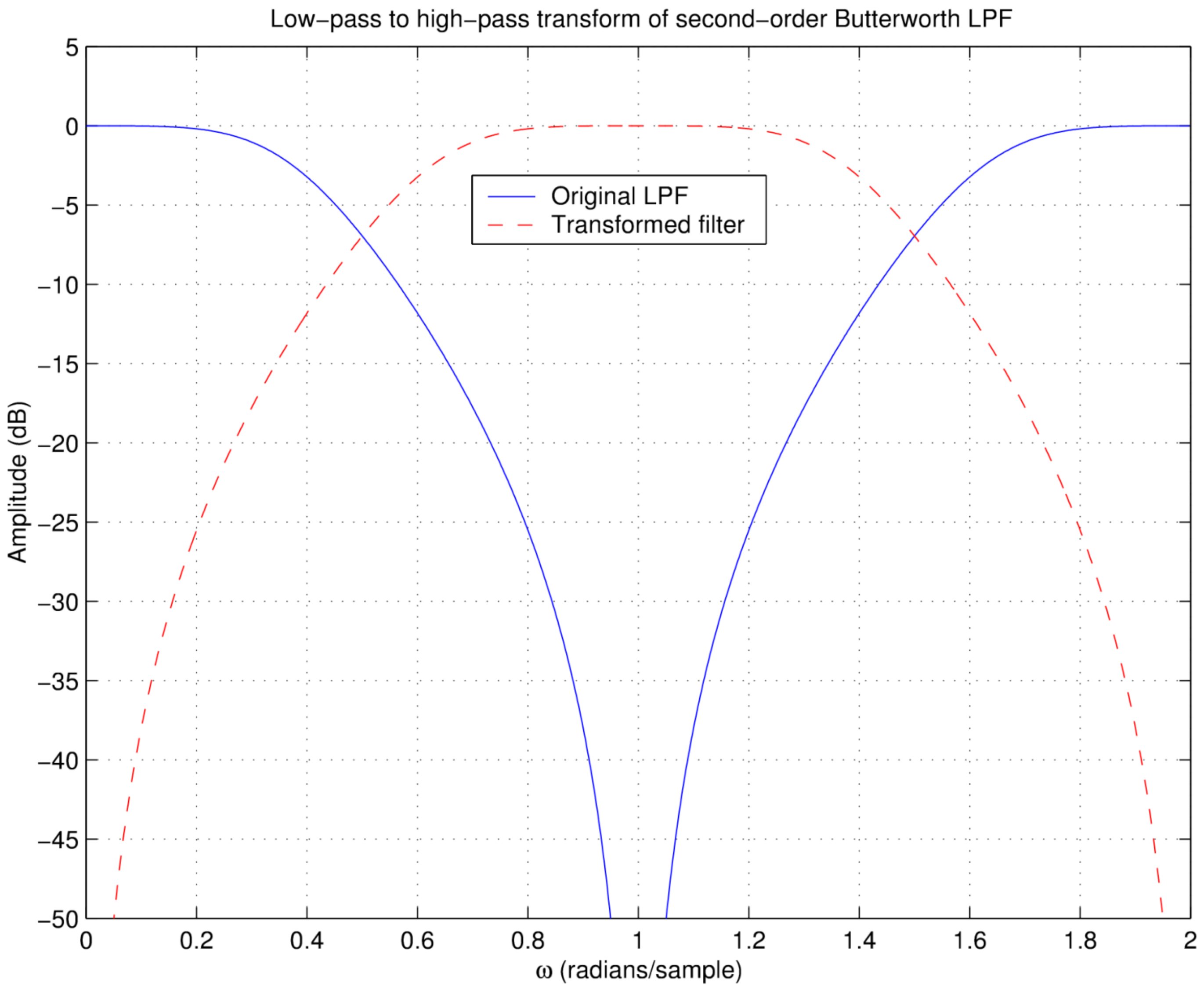


Butterworth LPF to HPF example

$$H_{\mathrm{LPF}}(z) = \frac{0.201 + 0.401z^{-1} + 0.201z^{-2}}{1 - 0.397z^{-1} + 0.2z^{-2}} \quad \Rightarrow \quad H_{\mathrm{trans}}(z)\frac{0.201 - 0.401z^{-1} + 0.201z^{-2}}{1 + 0.397z^{-1} + 0.2z^{-2}}$$

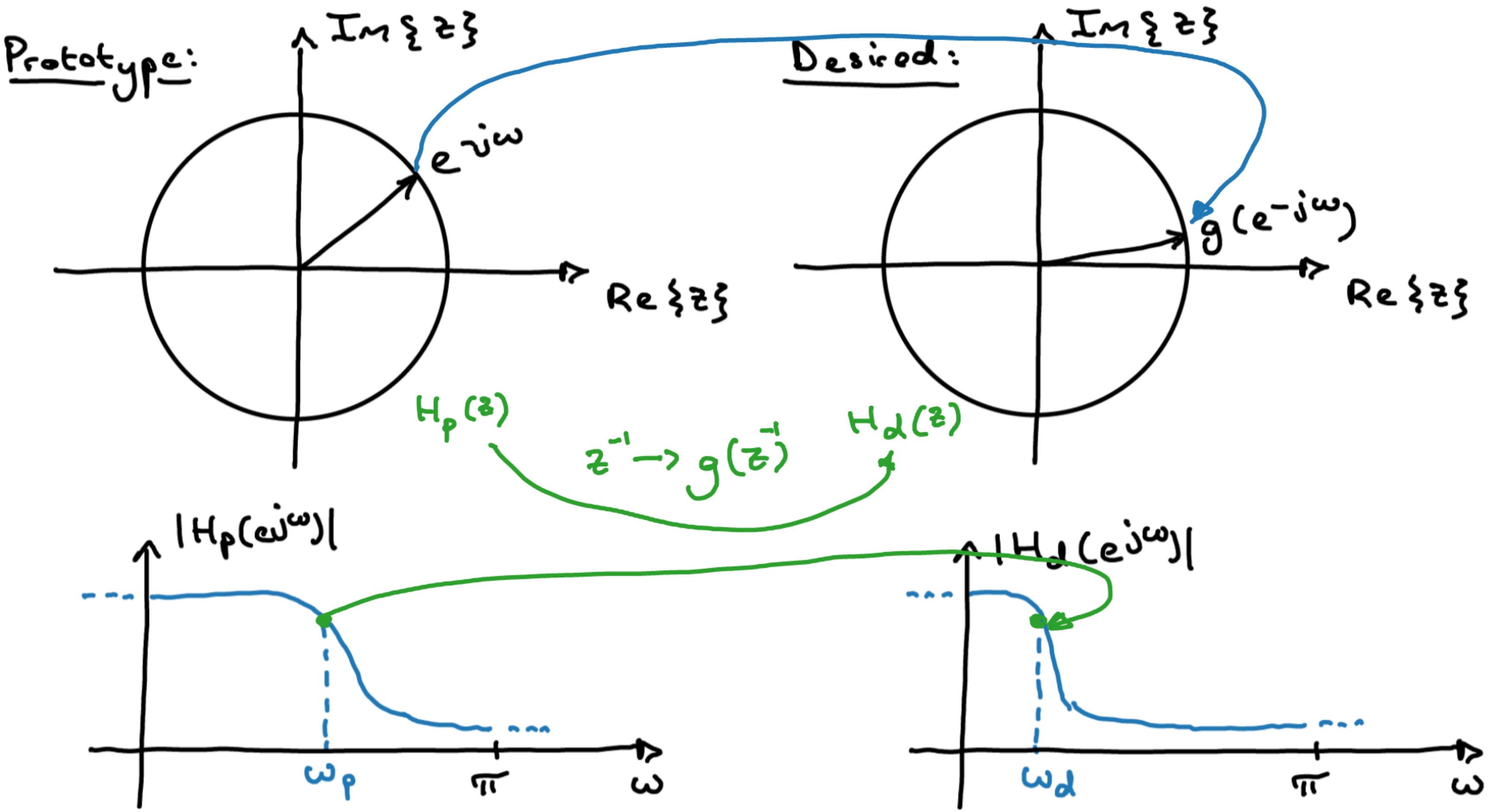






Low−pass to high−pass transform of second−order Butterworth LPF
Original LPF
Transformed filter
Amplitude (dB)
ω (radians/sample)
5
0
−5
−10
−15
−20
−25
−30
−35
−40
−45
−50
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

General frequency transforms



$z^{-1} \longrightarrow g(z^{-1})$

Prototype $\longrightarrow$ Desired

Setting $z = e^{j\omega}$:

$$e^{-j\omega} \to g(e^{-j\omega})$$

$g(z^{-1})$

$|g(e^{-j\omega})| = 1$

Points on the unit circle must map to points on the unit circle:

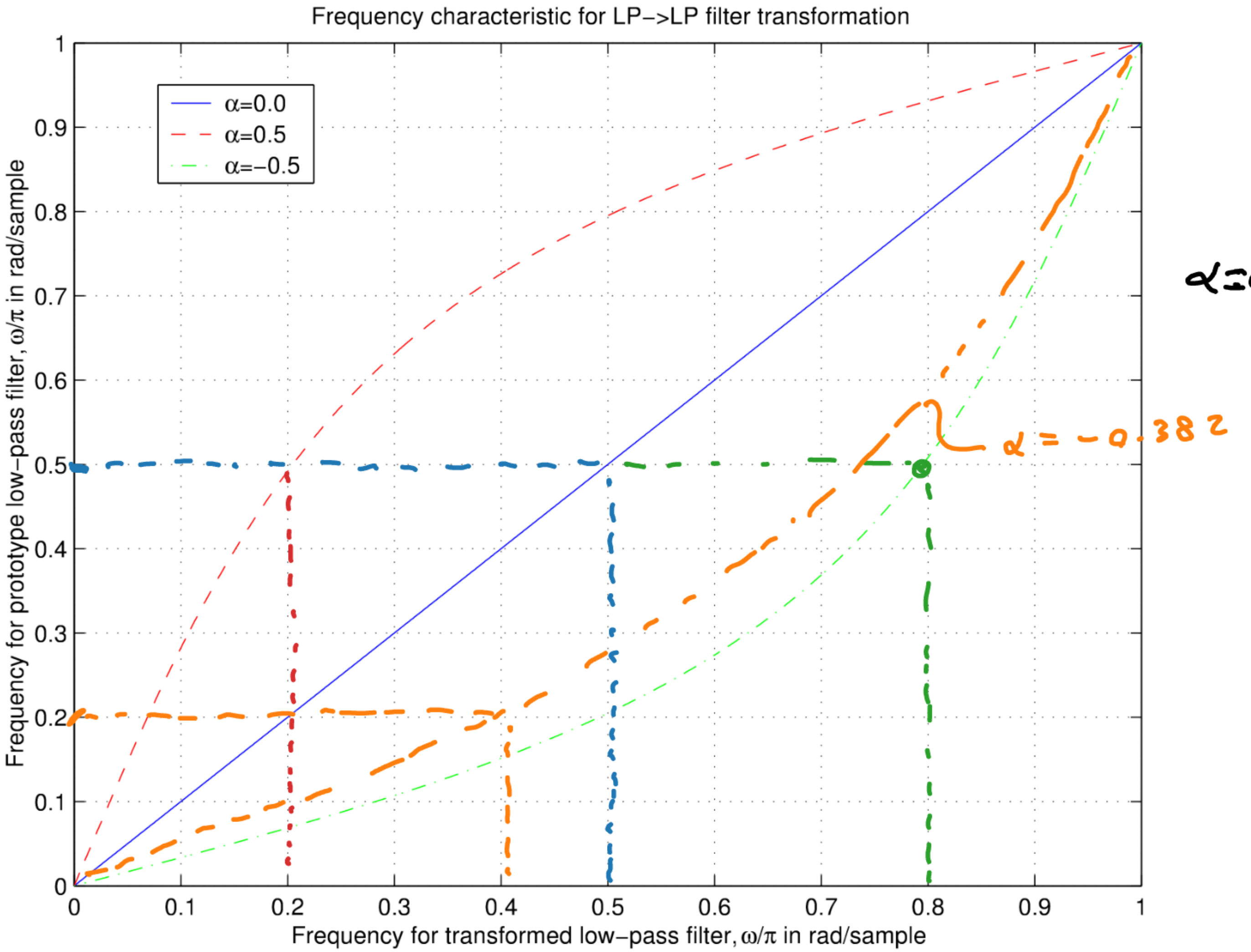
$$|g(e^{-j\omega})| = 1$$

Therefore $g(z^{-1})$ must be all-pass:

$$g(z^{-1}) = \pm \prod_{k=1}^{N} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

with $|\alpha_k| < 1$ for a stable mapping

Low-pass to low-pass transform



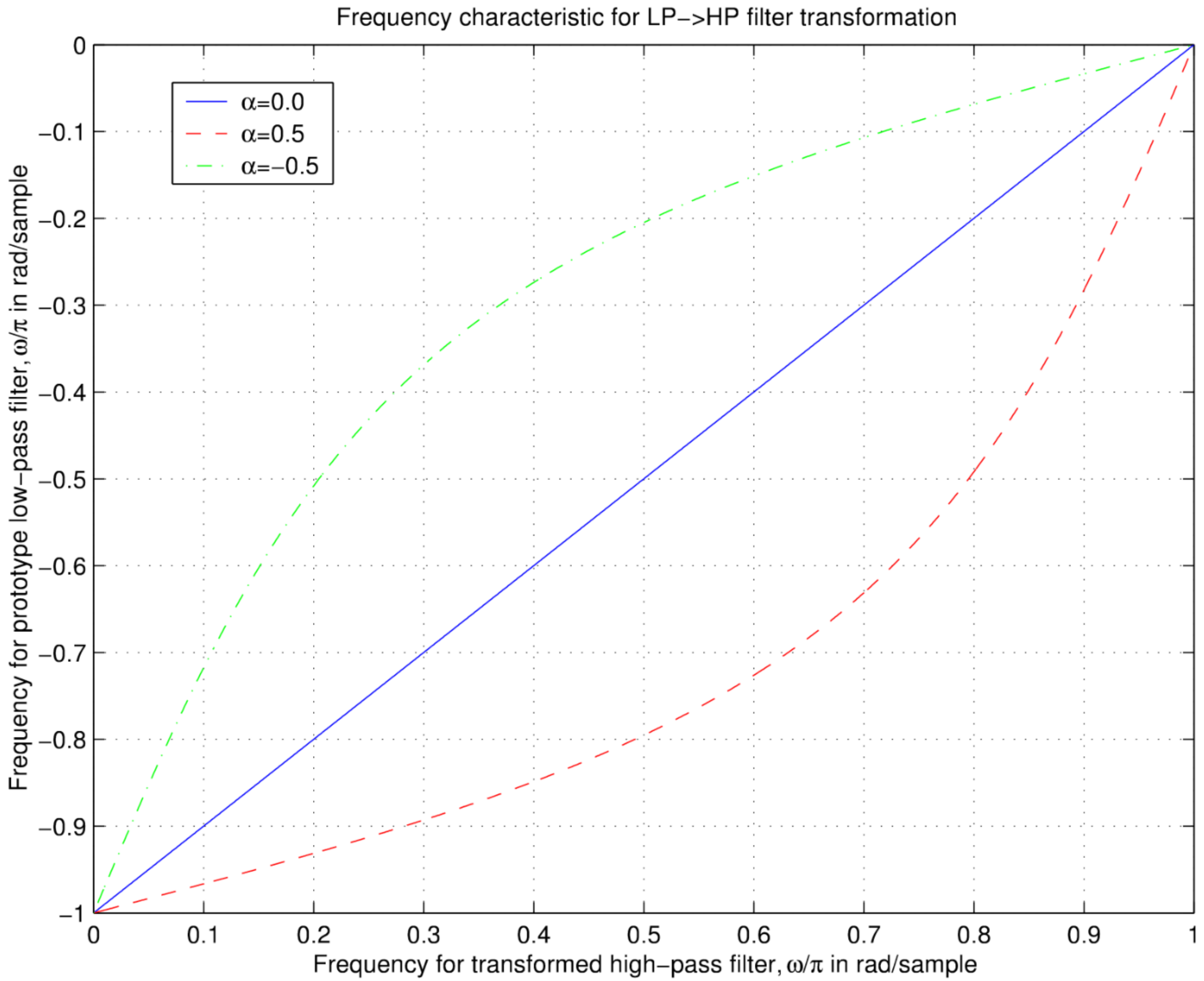
$$z^{-1} \to \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

$\alpha = 0: \; z^{-1} \to z^{-1}$

$$g(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

$$\alpha = \frac{\sin\left(\frac{\omega_p - \omega_d}{2}\right)}{\sin\left(\frac{\omega_p + \omega_d}{2}\right)}$$

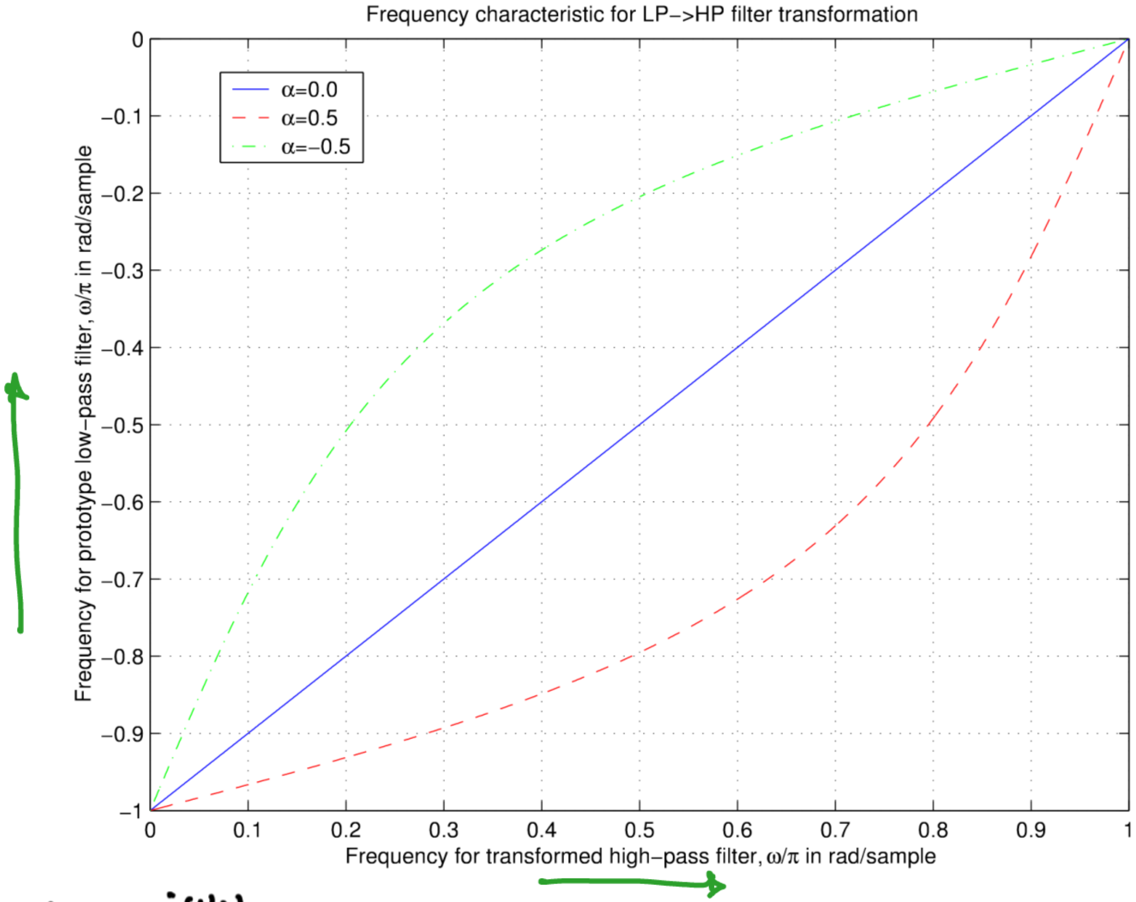
Low-pass to high-pass transform

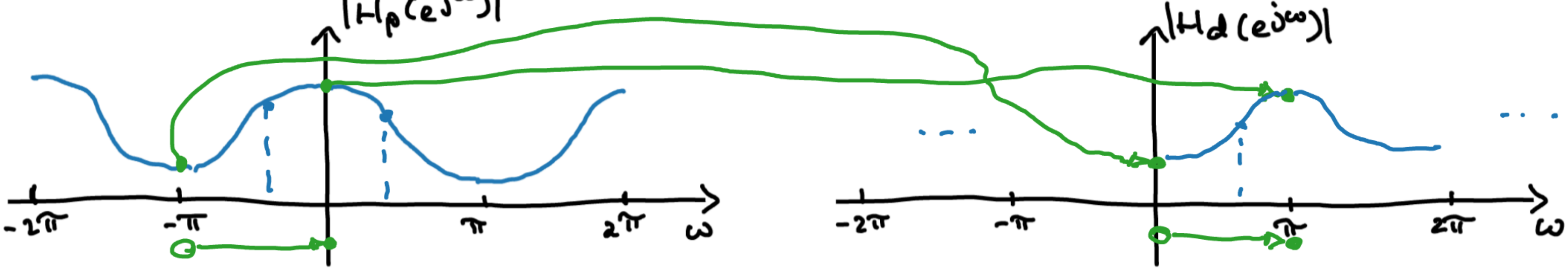


$\alpha = 0: \; z^{-1} \rightarrow -z^{-1}$

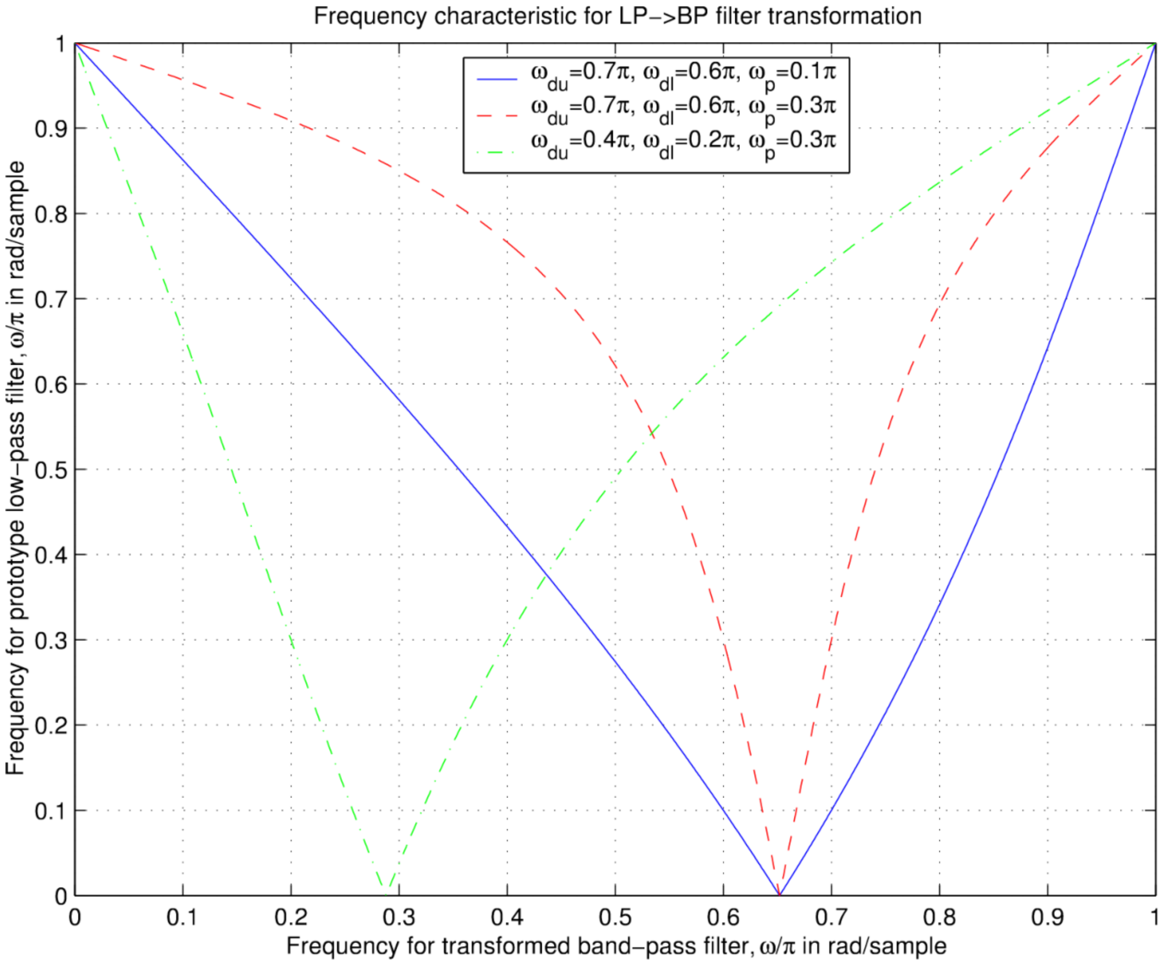
$$g(z^{-1}) = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$

$$\alpha = -\frac{\cos\left(\frac{\omega_p + \omega_d}{2}\right)}{\cos\left(\frac{\omega_p - \omega_d}{2}\right)}$$


Frequency characteristic for LP−>HP filter transformation
α=0.0
α=0.5
α=−0.5
Frequency for prototype low−pass filter, ω/π in rad/sample
Frequency for transformed high−pass filter, ω/π in rad/sample
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
−0.9
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


|Hp(ejω)|
|Hd(ejω)|
-2π
-π
π
2π
ω
0
-2π
-π
π
2π
ω
0

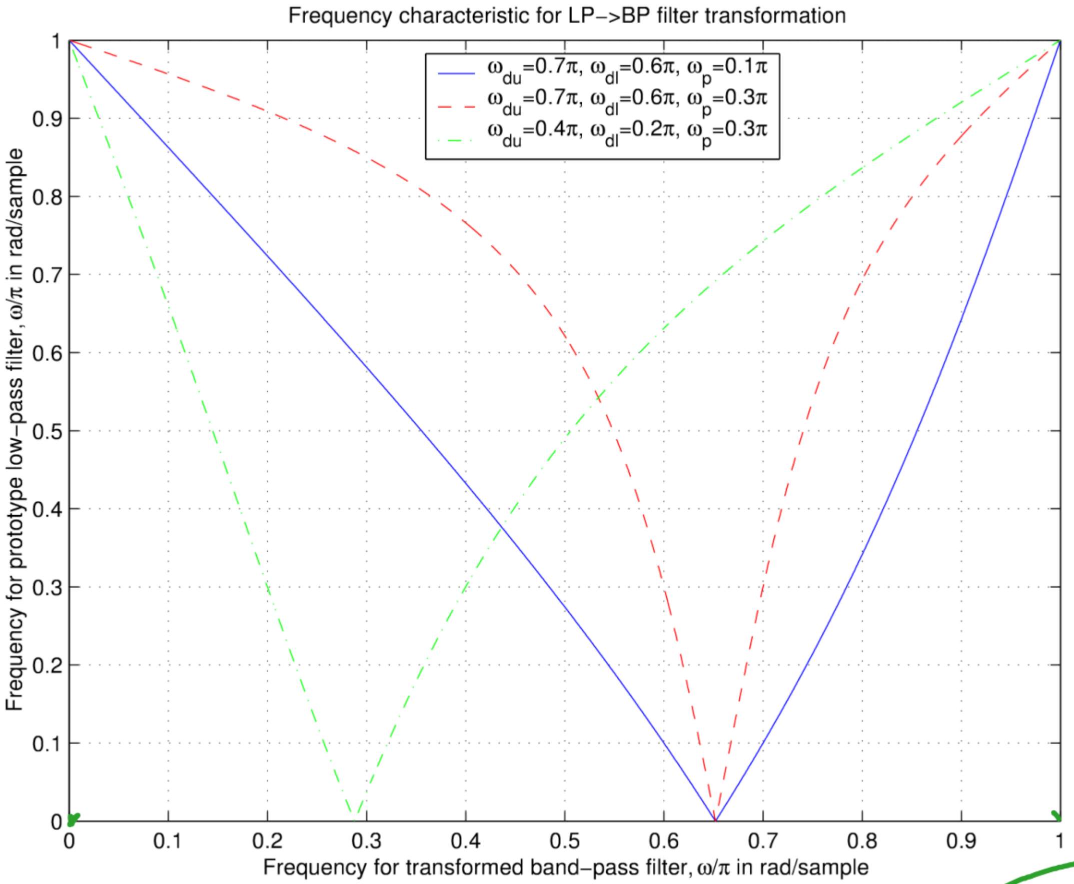
Low-pass to band-pass transform

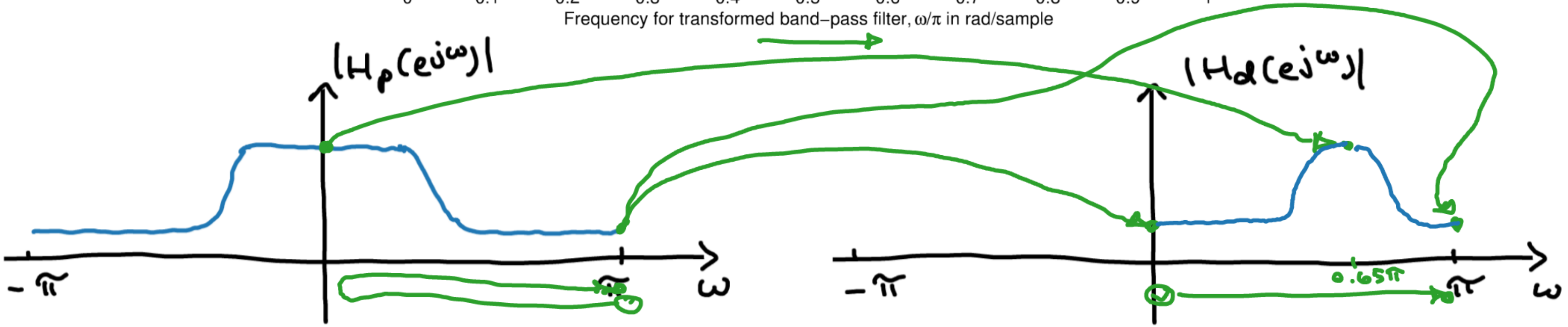


$$g(z^{-1}) = -\frac{\frac{k-1}{k+1} - \frac{2\alpha k}{k+1} z^{-1} + z^{-2}}{1 - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1} z^{-2}}$$

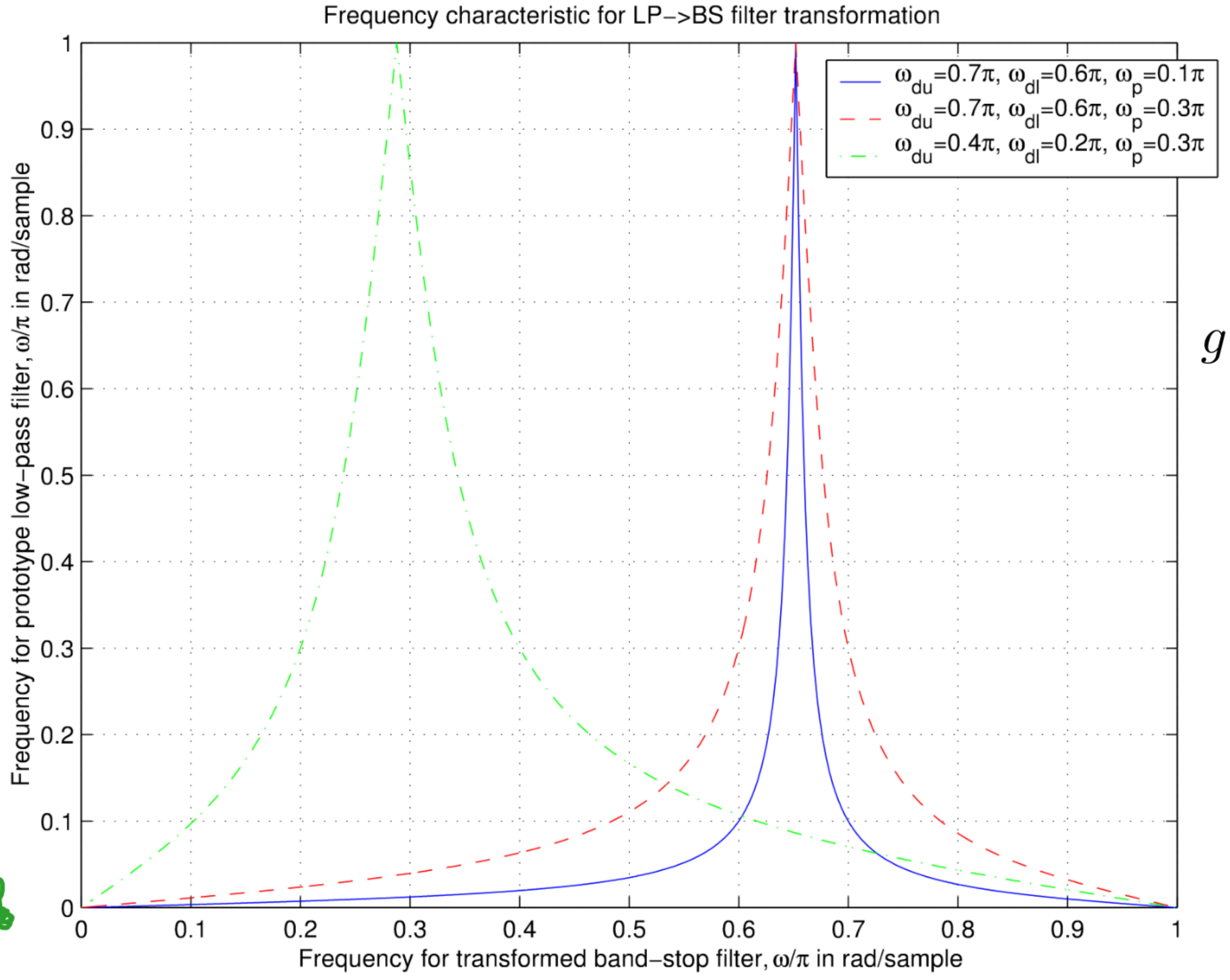
$$\alpha = \frac{\cos\left(\frac{\omega_{du}+\omega_{dl}}{2}\right)}{\cos\left(\frac{\omega_{du}-\omega_{dl}}{2}\right)}$$

$$k = \tan\left(\frac{\omega_p}{2}\right) / \tan\left(\frac{\omega_{du}-\omega_{dl}}{2}\right)$$


Frequency characteristic for LP−>BP filter transformation
ω_du=0.7π, ω_dl=0.6π, ω_p=0.1π
ω_du=0.7π, ω_dl=0.6π, ω_p=0.3π
ω_du=0.4π, ω_dl=0.2π, ω_p=0.3π
Frequency for prototype low−pass filter, ω/π in rad/sample
Frequency for transformed band−pass filter, ω/π in rad/sample
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


|Hp(e^jω)|
-π
π
ω
|Hd(e^jω)|
-π
0.65π
π
ω

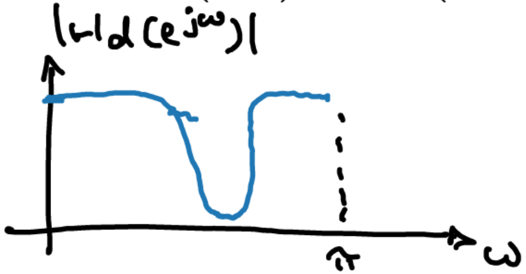
Low-pass to band-stop transform



$$g(z^{-1}) = -\frac{\frac{1-k}{1+k} - \frac{2\alpha}{k+1}z^{-1} + z^{-2}}{1 - \frac{2\alpha}{k+1}z^{-1} + \frac{1-k}{1+k}z^{-2}}$$

$$\alpha = \frac{\cos\left(\frac{\omega_{du}+\omega_{dl}}{2}\right)}{\cos\left(\frac{\omega_{du}-\omega_{dl}}{2}\right)}$$

$$k = \tan\left(\frac{\omega_p}{2}\right)\tan\left(\frac{\omega_{du}-\omega_{dl}}{2}\right)$$



Low-pass to low-pass transform example

Below is a prototype LPF with a -3 dB cut-off at $\omega_p = 0.2\pi$ rad/sample. Transform this filter into a LPF with a cut-off at $\omega_d = 0.4\pi$ rad/sample.

$$H_p(z) = \frac{0.106(1 + 2z^{-1} + z^{-2})}{1.565 - 1.789z^{-1} + 0.646z^{-2}}$$

$$H_d(z) = H_p(z)\Big|_{z^{-1} \to \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}} = \frac{b_0 + b_1\left(\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}\right) + b_2\left(\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}\right)^2}{a_0 + a_1\left(\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}\right) + a_2\left(\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}\right)^2}$$

$$\alpha = \frac{\sin\left(\frac{\omega_p - \omega_d}{2}\right)}{\sin\left(\frac{\omega_p + \omega_d}{2}\right)} = \frac{\sin\left(\frac{0.2\pi - 0.4\pi}{2}\right)}{\sin\left(\frac{0.2\pi + 0.4\pi}{2}\right)} = -0.382$$

Answer: $H_d(z) = \dfrac{0.2074 + 0.4149z^{-1} + 0.2074z^{-2}}{1 - 0.3699z^{-1} + 0.1957z^{-2}}$

Low-pass to high-pass transform example

Below is a prototype LPF with a -3 dB cut-off at $\omega_p = 0.2\pi$ rad/sample. Transform this filter into a HPF with a cut-off at $\omega_d = 0.4\pi$ rad/sample.

$$H_p(z) = \frac{0.106(1 + 2z^{-1} + z^{-2})}{1.565 - 1.789z^{-1} + 0.646z^{-2}}$$

$$z^{-1} \longrightarrow -\frac{z^{-1} - \alpha}{1 + \alpha z^{-1}}$$

$$\alpha = -\frac{\cos\left(\frac{\omega_p + \omega_d}{2}\right)}{\cos\left(\frac{\omega_p - \omega_d}{2}\right)} = -0.618$$

$$H_d(z) = H_p(z)\Bigg|_{z^{-1} \longrightarrow -\frac{z^{-1} + 0.618}{1 - 0.618z^{-1}}}$$

Answer: $H_d(z) = \dfrac{0.278 - 0.555z^{-1} + 0.278z^{-2}}{0.706 - 0.261z^{-1} + 0.138z^{-2}}$