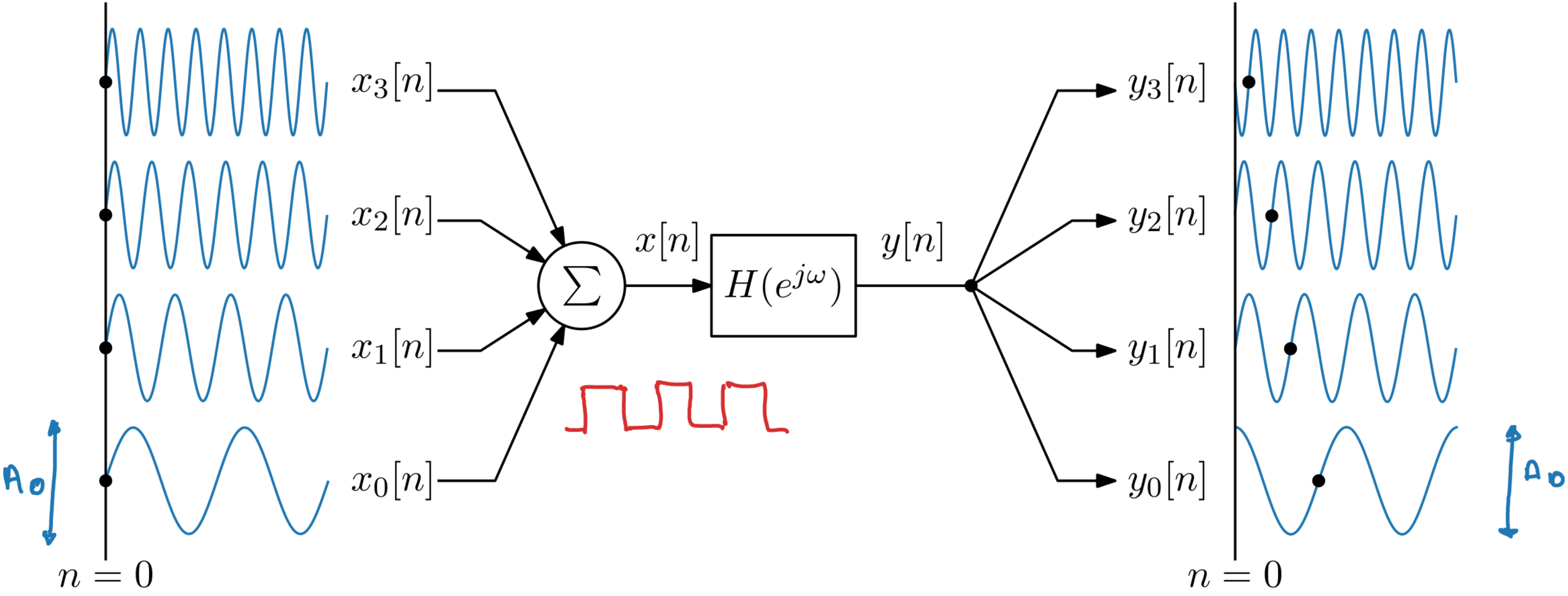
Filter phase characteristics

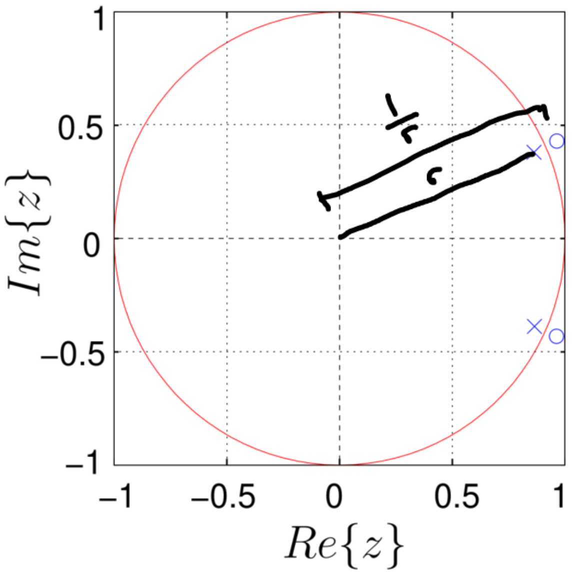
All-pass filters and linear phase

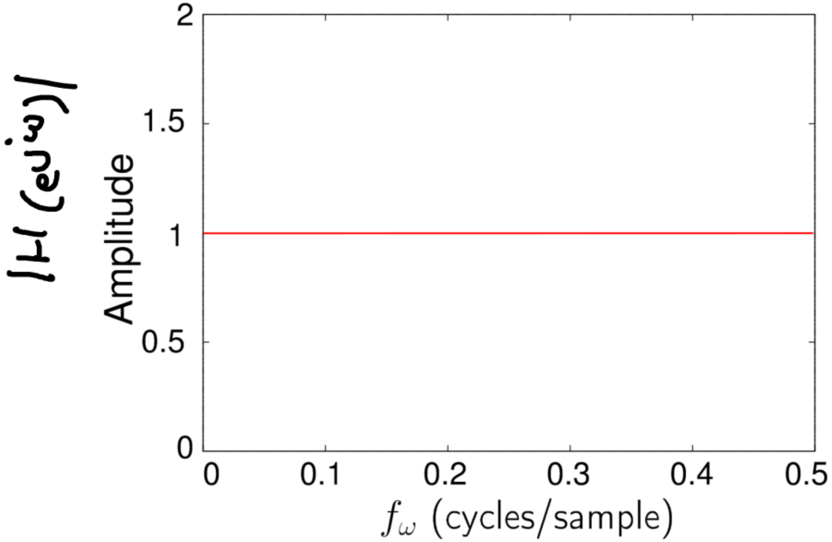
Herman Kamper

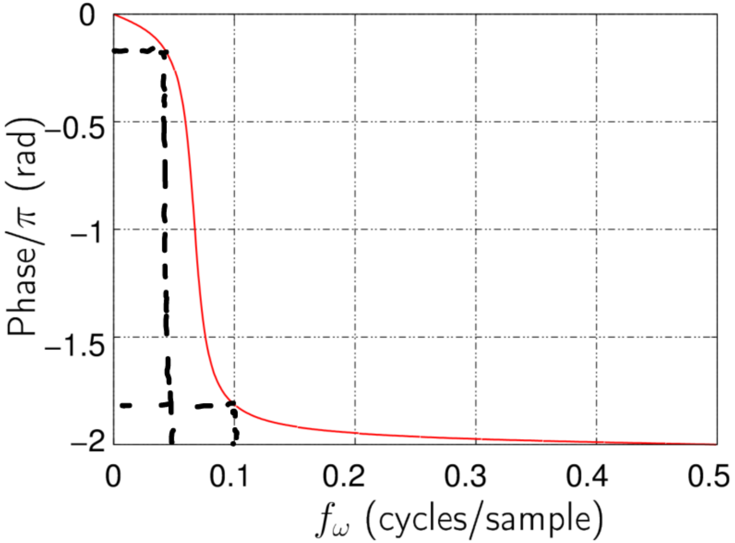
Filter phase characteristics



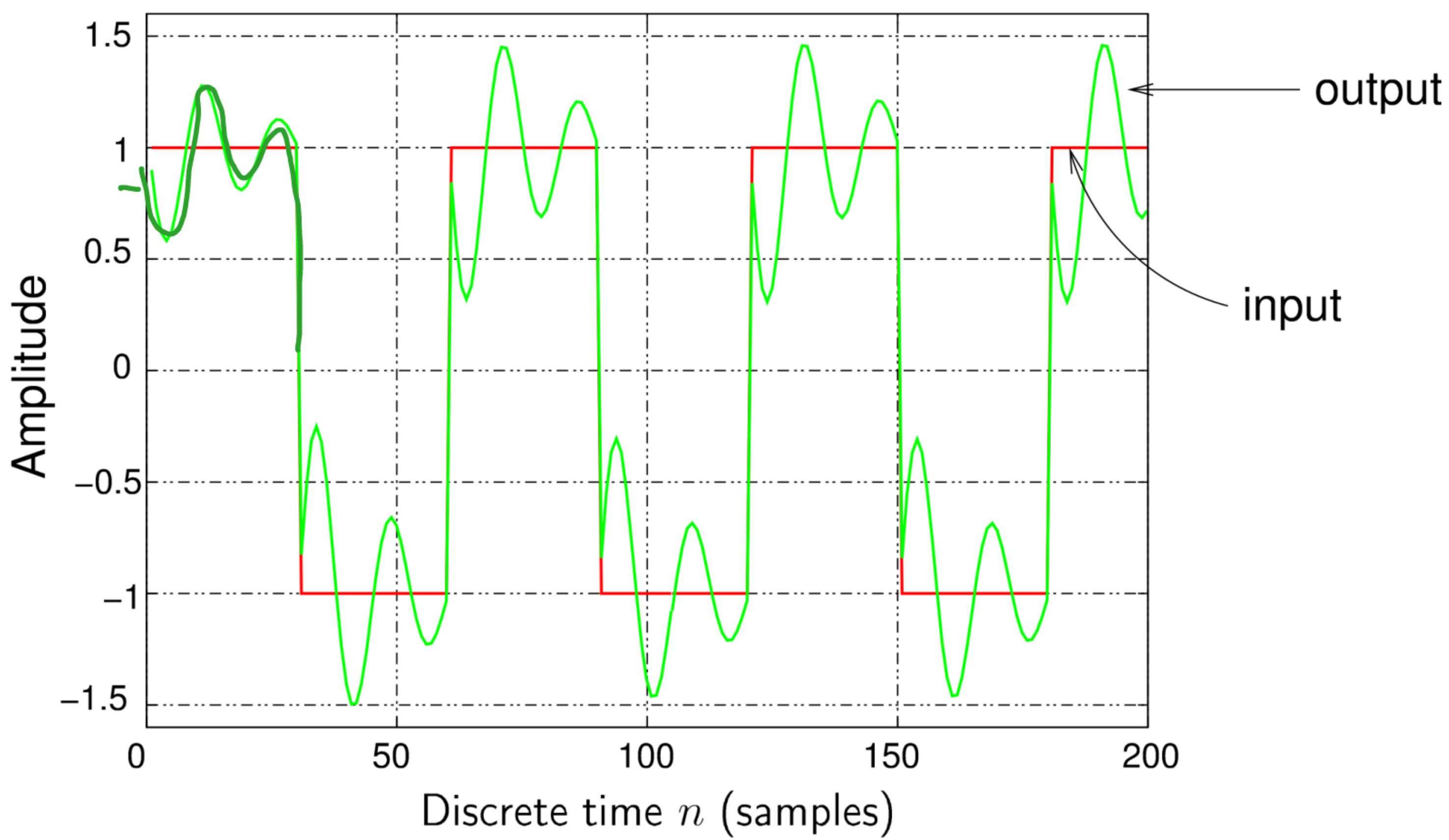
$$H(z) = \frac{0.9 - \sqrt{3}z^{-1} + z^{-2}}{1 - \sqrt{3}z^{-1} + 0.9z^{-2}}$$







All-pass filter input and output:



Linear phase

$$x[n] = \sum_{i=0}^{K-1} x_i[n]$$

What we want:

$$y_i[n] = A_i \, x_i[n - n_0] \quad \cdots \text{①}$$

and n_0 should be the same for all i

$$x_i[n] = e^{j\omega_i n} \quad \cdots \text{②}$$

then $y_i[n] = |H(e^{j\omega_i})| \cdot e^{j(\omega_i n + \angle H(e^{j\omega_i}))}$

$$= A_i \, e^{j(\omega_i n + \phi_i)}$$

For ① to be true:

$$x_i[n - n_0] = e^{j(\omega_i n + \phi_i)}$$

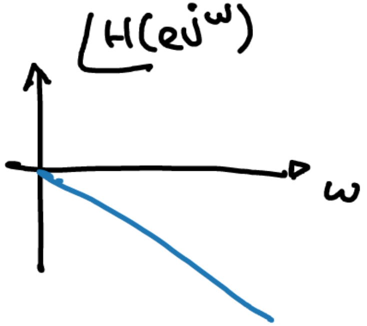
②

$$e^{j\omega_i (n - n_0)} = e^{j(\omega_i n + \phi_i)}$$

$$e^{j\omega_i n - j\omega_i n_0} = e^{j\omega_i n + j\phi_i}$$

$$\phi_i = -\omega_i n_0$$

$$\underline{\angle H(e^{j\omega})} = -\omega n_0$$



All-pass filters

$$|H(e^{j\omega})| = 1 \text{ for all } \omega$$

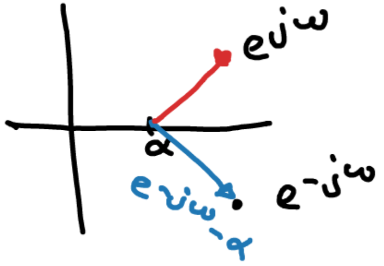
$A(z) = z^2 + z - 2 = (z+2)(z-1)$

$x = z^{-1}$

$A'(z) = z^{-2} + z^{-1} - 2 = x^2 + x - 2 = (x+2)(x-1)$
$= (z^{-1}+2)(z^{-1}-1)$

$$A(z) = z^N + a_{N-1}z^{N-1} + a_{N-2}z^{N-2} + \ldots + a_1 z + a_0$$

$$A'(z) = z^{-N} + a_{N-1}z^{-(N-1)} + a_{N-2}z^{-(N-2)} + \ldots + a_1 z^{-1} + a_0$$



For every factor $(z-\alpha)$ in $A(z)$
we have $(z^{-1}-\alpha)$ in $A'(z)$

For real α: $|z^{-1}-\alpha|_{z=e^{j\omega}} = |e^{-j\omega}-\alpha| = |e^{j\omega}-\alpha| = |z-\alpha|_{z=e^{j\omega}}$

For complex factors: $|z^{-1}-\alpha||z^{-1}-\alpha^*|_{z=e^{j\omega}} = |z-\alpha||z-\alpha^*|_{z=e^{j\omega}}$

$$\therefore \left|\frac{A'(z)}{A(z)}\right|_{z=e^{j\omega}} = 1 \quad \text{for all } \omega$$

$$H(z) = \frac{A'(z)}{A(z)/z^N} = \frac{z^{-N} + a_{N-1}z^{-(N-1)} + a_{N-2}z^{-(N-2)} + \ldots + a_1 z^{-1} + a_0}{1 + a_{N-1}z^{-1} + a_{N-2}z^{-2} + \ldots + a_1 z^{-(N-1)} + a_0 z^{-N}} = \frac{A'(z)}{A(z)/z^N}$$

$$= \prod_{k=0}^{N_r} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=0}^{N_c} \boxed{\frac{(z^{-1} - \beta_k)}{(1 - \beta_k z^{-1})}} \frac{(z^{-1} - \beta_k^*)}{(1 - \beta_k^* z^{-1})}$$

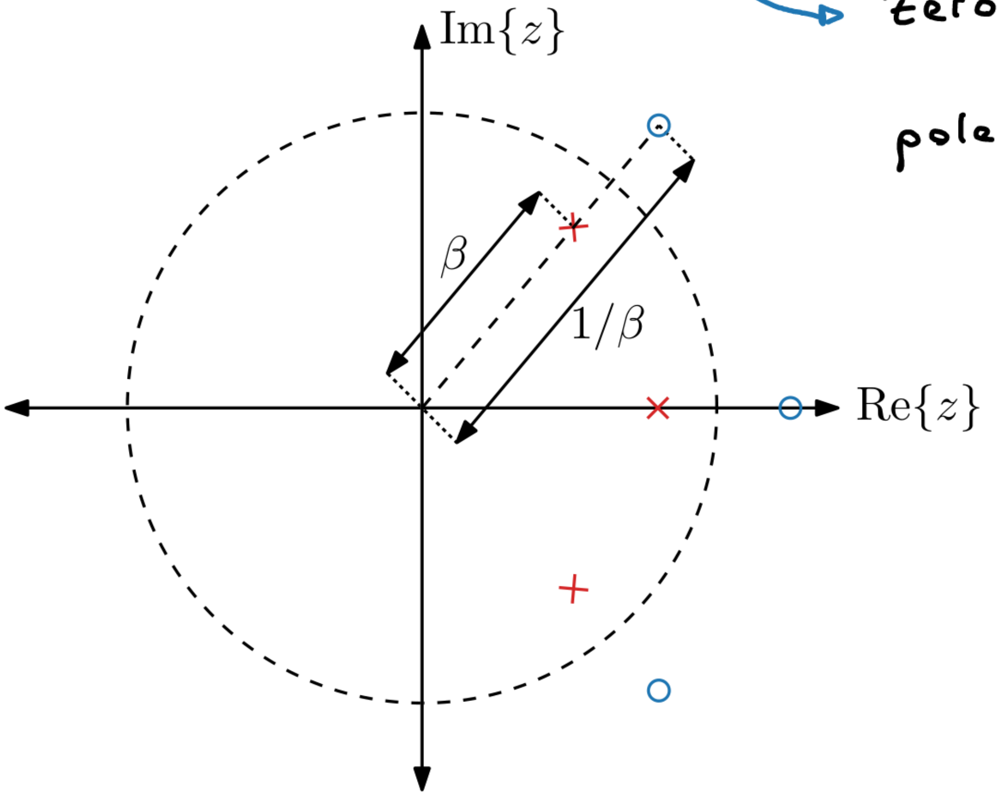
zero: $z^{-1} - \beta_k = 0$

zero $= \frac{1}{\beta_k}$

pole: $1 - \beta_k z^{-1}$

$= 1 - \frac{\beta_k}{z}$

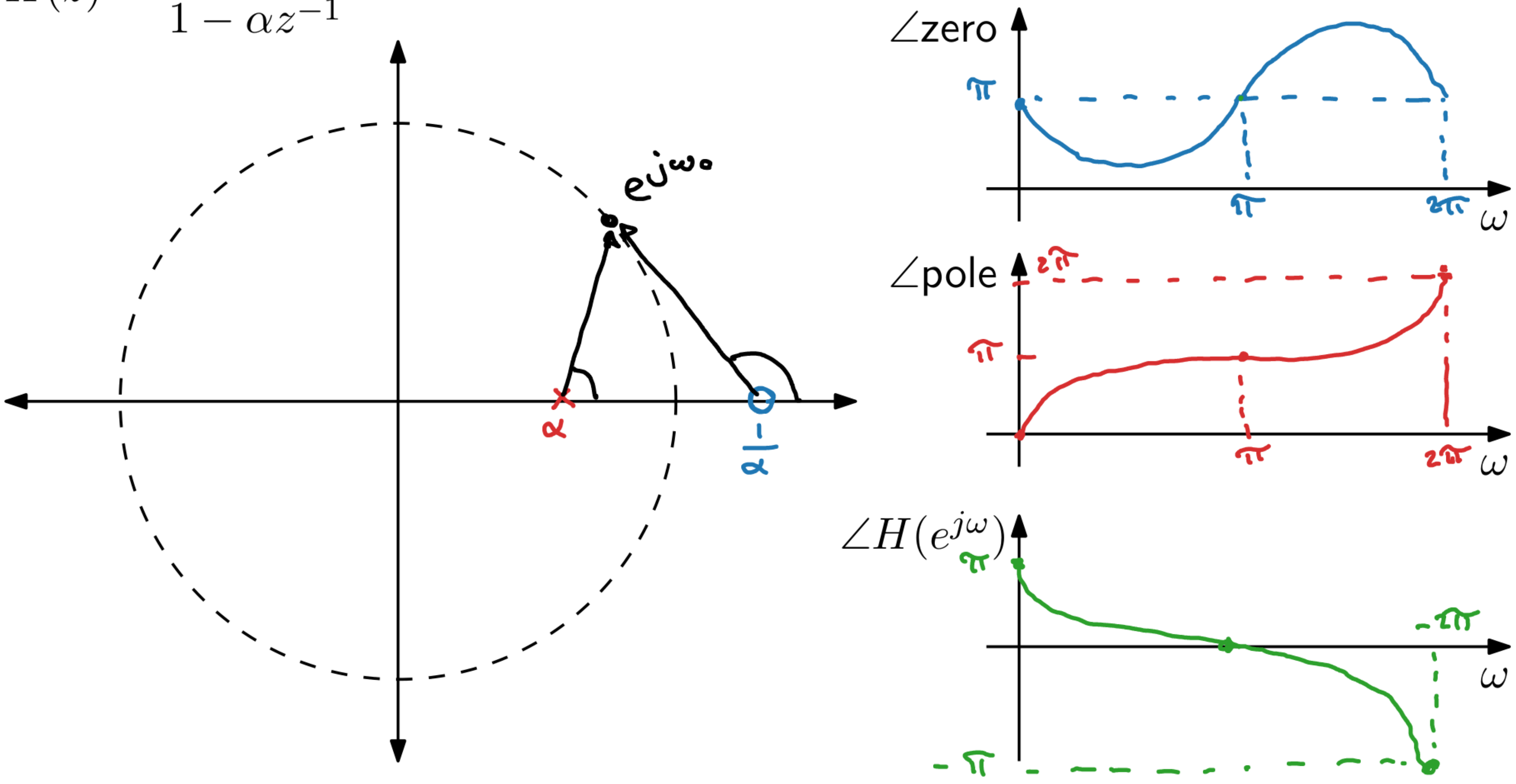
Pole $= \beta_k$

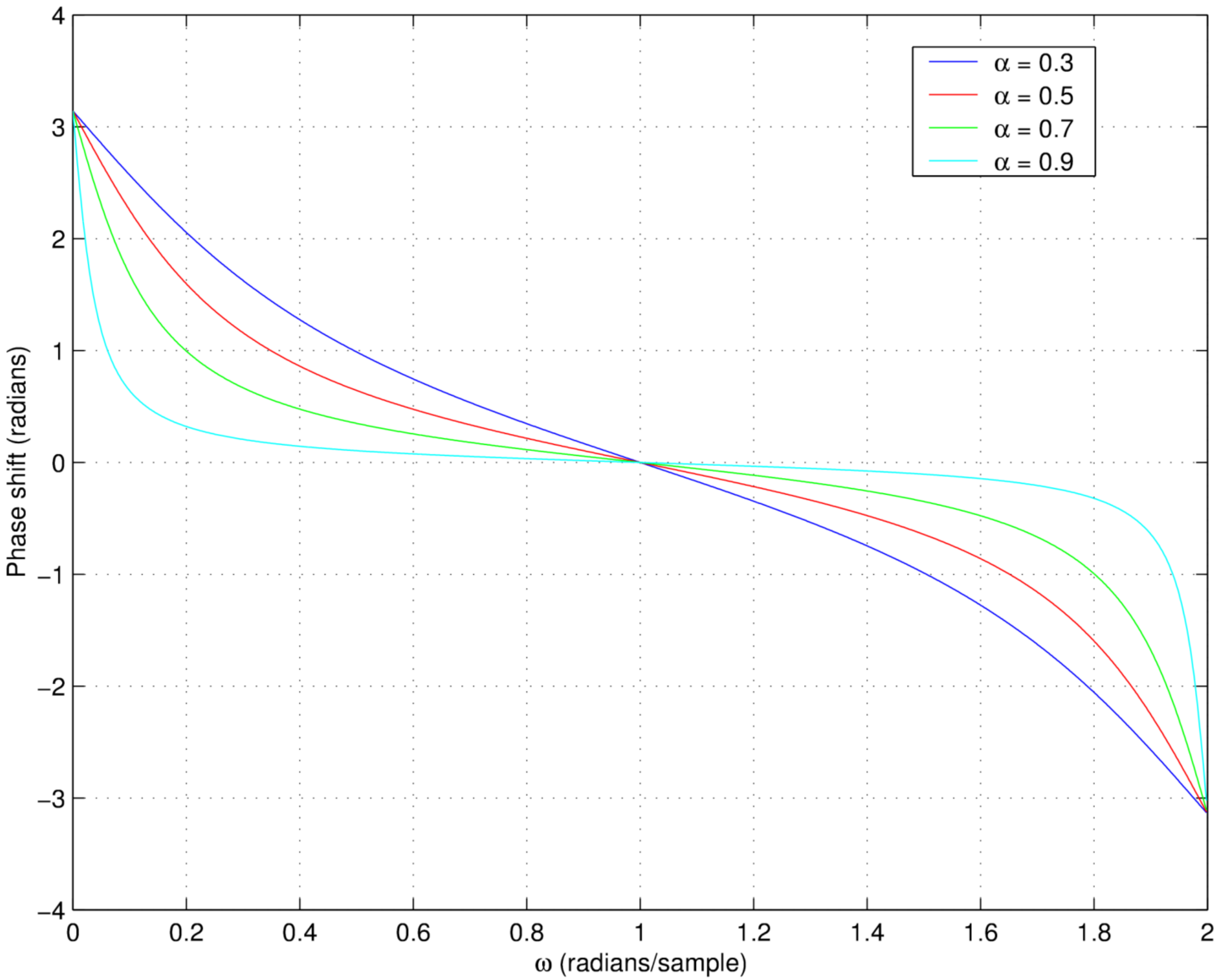


All-pass filter example

$$H(z) = \frac{1 - \frac{1}{\alpha} z^{-1}}{1 - \alpha z^{-1}}$$

$\angle H(e^{j\omega}) = \angle \text{zeros} - \angle \text{poles}$





α = 0.3
α = 0.5
α = 0.7
α = 0.9
Phase shift (radians)
ω (radians/sample)
4
3
2
1
0
−1
−2
−3
−4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2