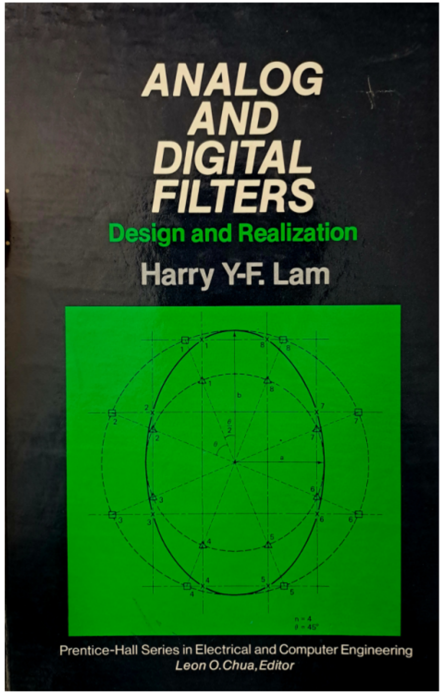
Infinite impulse response (IIR) filter design

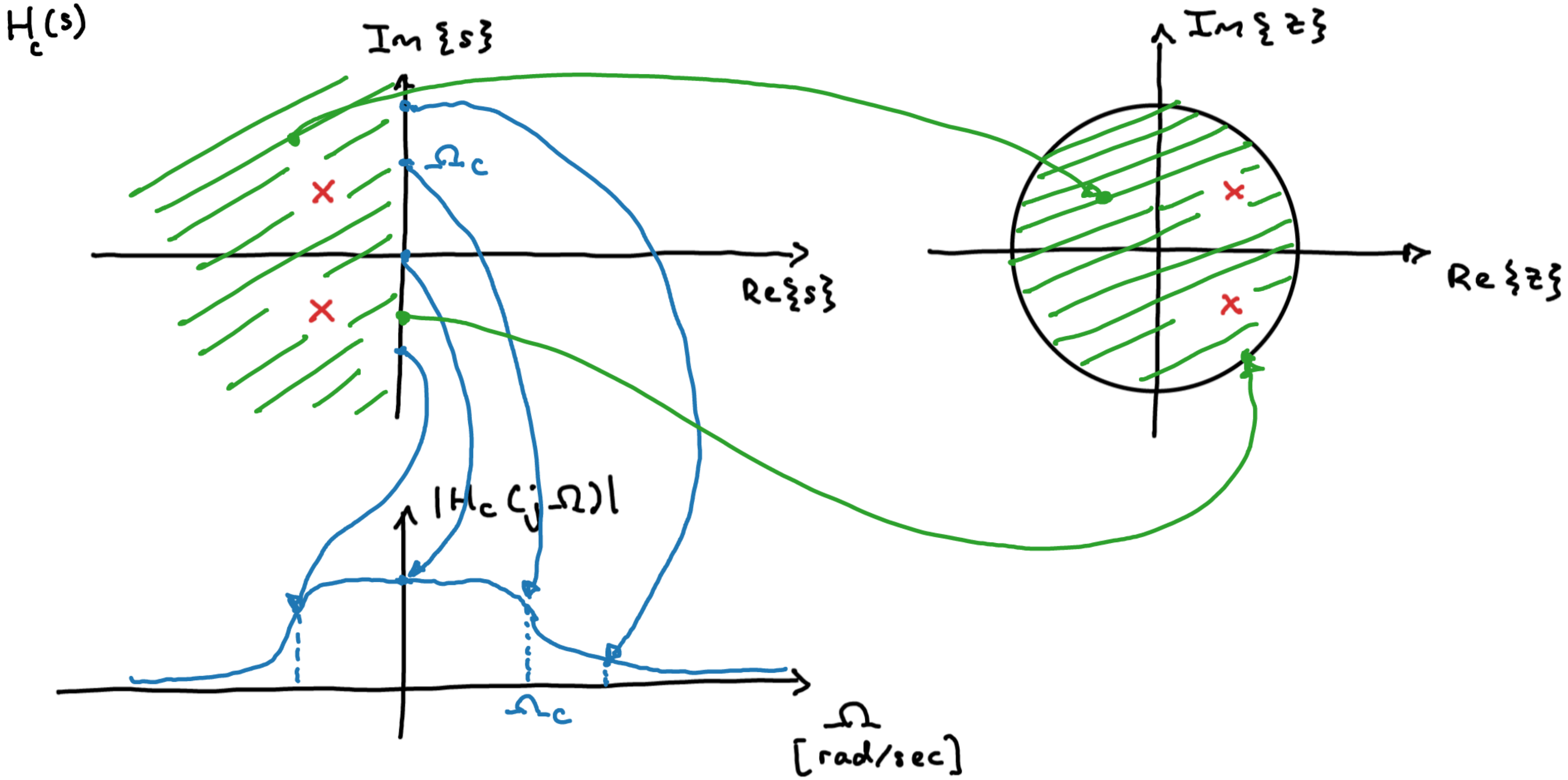
Herman Kamper

Discrete filter design methods

- Place poles and zeros
- Hack the ideal impulse response to make it realisable (FIR)
- Convert continuous filters to discrete filters (IIR)



Mapping the s-plane to the z-plane:



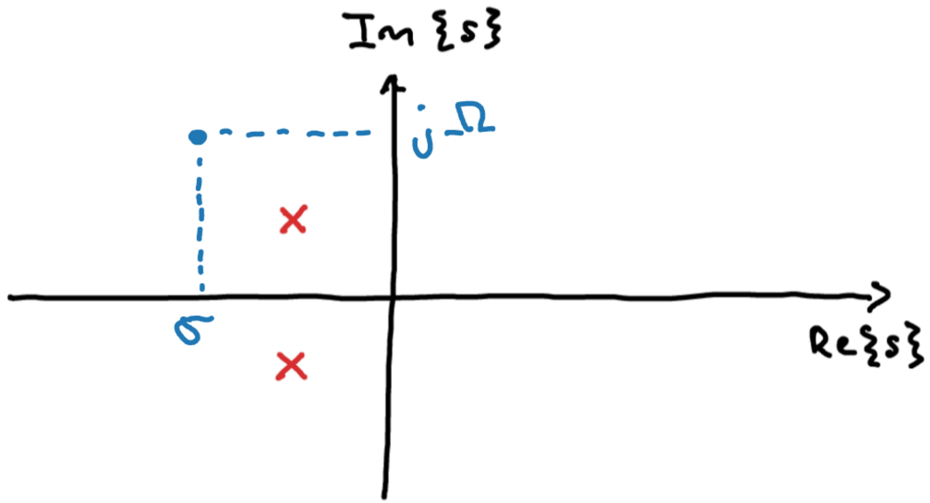
Bilinear transform

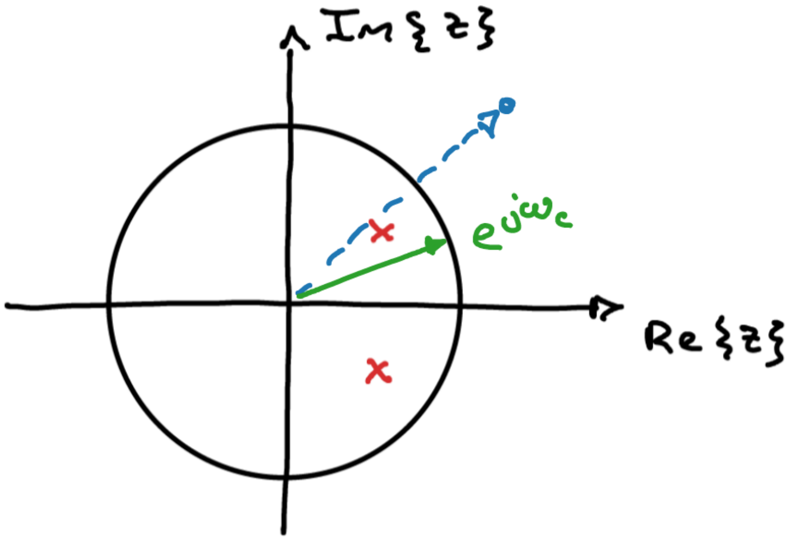
$$s = \frac{1}{K}\frac{1-z^{-1}}{1+z^{-1}} \quad \text{and} \quad z = \frac{1+Ks}{1-Ks} \quad \text{with } K > 0$$

Let $s = \sigma + j\Omega$:

$$z = \frac{1+K(\sigma+j\Omega)}{1-K(\sigma+j\Omega)} = \frac{1+K\sigma+jK\Omega}{1-K\sigma-jK\Omega} \Rightarrow |z|^2 = \frac{(1+K\sigma)^2+K^2\Omega^2}{(1-K\sigma)^2+K^2\Omega^2}$$

- What is $|z|$ if $\sigma = 0$? $|z| = 1$
- What is $|z|$ if $\sigma < 0$? $|z| < 1$





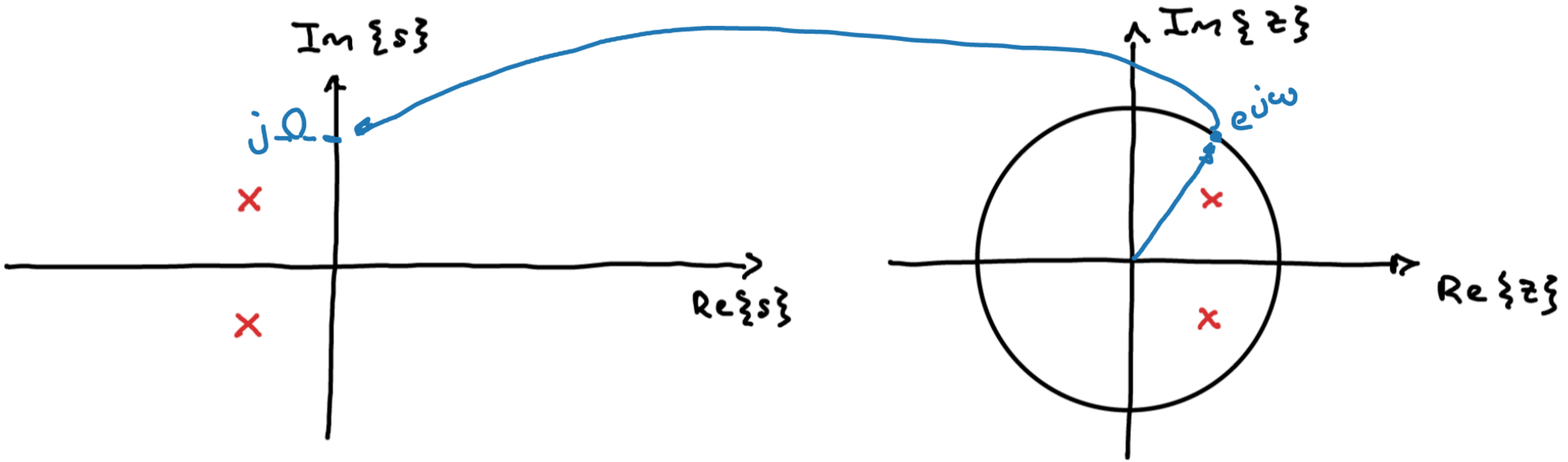
$$s = \frac{1}{K}\frac{1-z^{-1}}{1+z^{-1}} \quad \text{and} \quad z = \frac{1+Ks}{1-Ks}$$

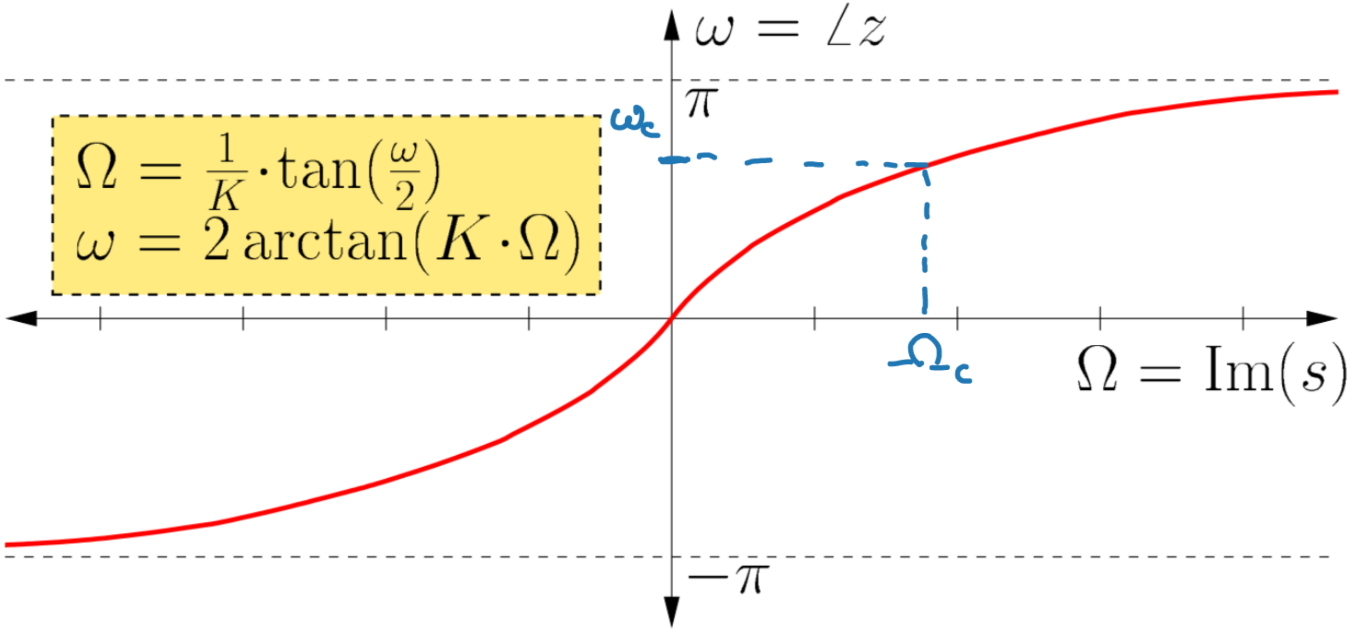
Let $z = e^{j\omega}$:

$$s = \frac{1}{K}\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{1}{K}\frac{\cancel{e^{-j\omega/2}}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}{\cancel{e^{-j\omega/2}}\left(e^{j\omega/2}+e^{-j\omega/2}\right)}\cdot\frac{2}{2}\cdot\frac{2j}{2j}$$

$$= \frac{1}{K}\frac{\sin(\omega/2)}{\cos(\omega/2)}\cdot\frac{\cancel{2}j}{\cancel{2}} = \frac{j}{K}\frac{\sin(\omega/2)}{\cos(\omega/2)} = \frac{j}{K}\tan\left(\frac{\omega}{2}\right)$$

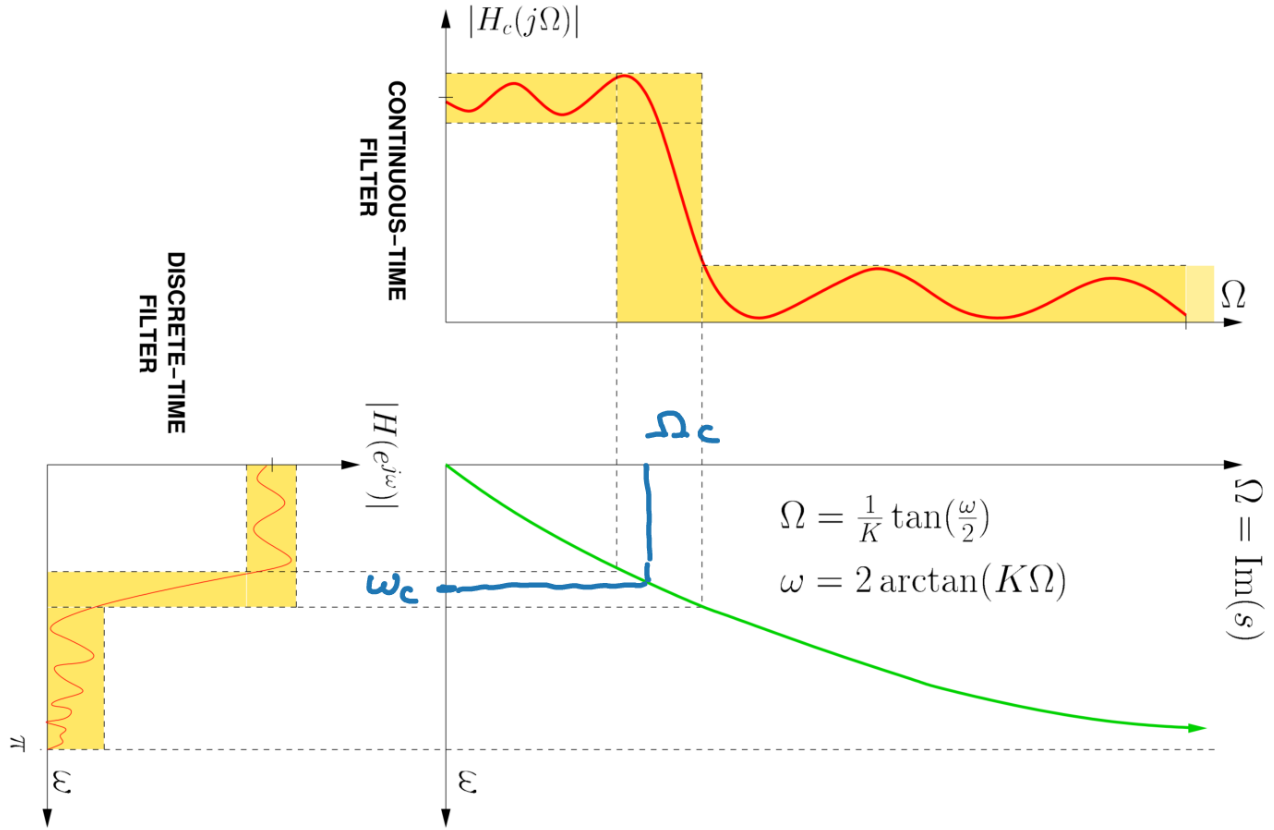
$$\therefore \Omega = \frac{1}{K}\tan\left(\frac{\omega}{2}\right) \quad \text{and} \quad \omega = 2\arctan(\Omega K)$$





Maps: $\Omega \in (-\infty, \infty)$ to $\omega \in (-\pi, \pi)$

Continuous to discrete filter design procedure



1. Specification in discrete time
2. Pre-warp specification frequencies $\Omega = \frac{1}{K}\tan\left(\frac{\omega}{2}\right)$
3. Design continuous-time filter to pre-warped specification $H_c(s)$
4. Substitute bilinear transform $s = \frac{1}{K}\frac{1-z^{-1}}{1+z^{-1}}$

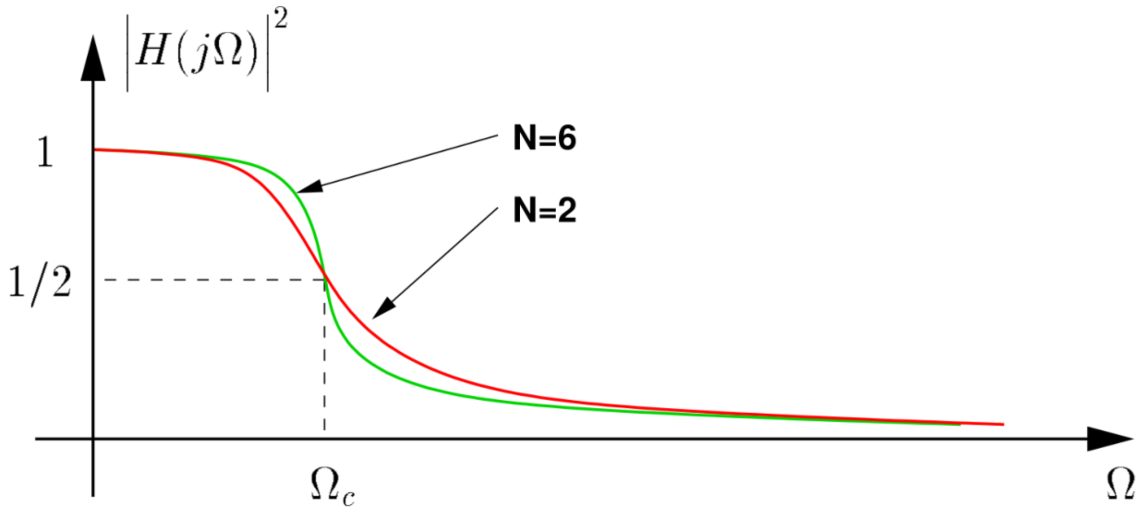
Continuous-time Butterworth filter

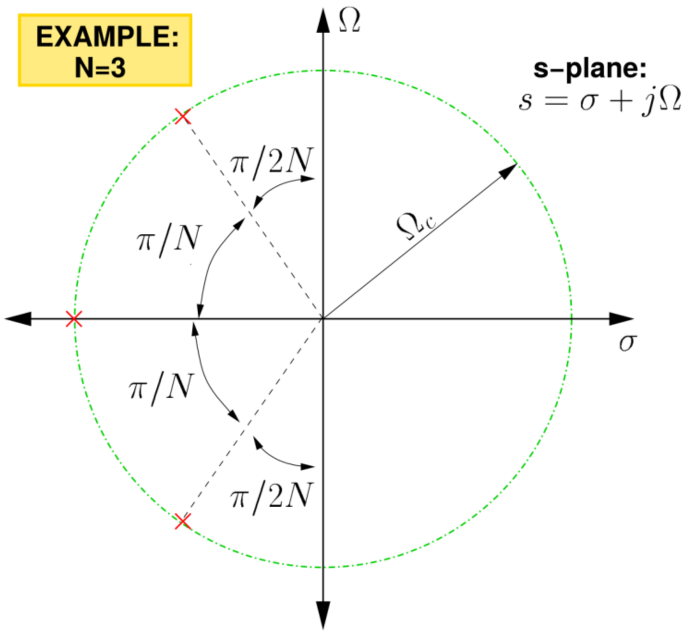
Butterworth LPF of order N has magnitude response:

$$|H_c(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

with N poles at

$$s_k = \Omega_c e^{j\pi \frac{N+1+2k}{2N}} \text{ for } k = 0, 1, \ldots, N-1$$





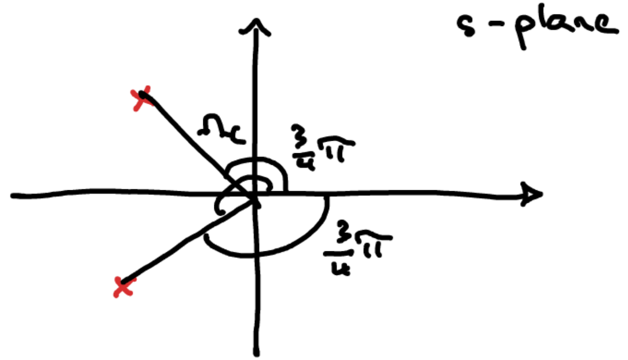
Butterworth discrete filter example

Use a second-order ($N=2$) analog Butterworth LPF as a prototype to design a discrete LPF with $\omega_c = 0.2\pi$ rad/sample.

1. Specification in discrete time $\omega_c = 0.2\pi$ rad/sample
2. Pre-warp specification frequencies $\Omega = \frac{1}{K}\tan\left(\frac{\omega}{2}\right)$
3. Design continuous-time filter to pre-warped specification

$K=1$

$$\Omega_c = \frac{1}{K}\tan\left(\frac{0.2\pi}{2}\right) = 0.325 \text{ rad/sec}$$



$N=2$: $\quad s_k = \Omega_c e^{j\pi\frac{N+1+2k}{2N}}$ for $k=0,1$

$$s_0 = \Omega_c \cdot e^{j\pi\frac{2+1+0}{2\cdot 2}} = \Omega_c e^{j\frac{3}{4}\pi}$$

$$s_1 = \Omega_c e^{j\frac{5}{4}\pi}$$

$c = \Omega_c^2$ (unity gain)

$$H_c(s) = \frac{c}{\left(s - \Omega_c e^{j\frac{3}{4}\pi}\right)\left(s - \Omega_c e^{j\frac{5}{4}\pi}\right)}$$

$$= \frac{\Omega_c^2}{s^2 + \sqrt{2}\,\Omega_c s + \Omega_c^2}$$

4. Substitute bilinear transform $s = \frac{1}{K}\frac{1-z^{-1}}{1+z^{-1}}$

$$H_c(s) = \frac{\Omega_c^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$

$$H(z) = H_c(s)\Big|_{s=\frac{1}{K}\frac{1-z^{-1}}{1+z^{-1}}}$$

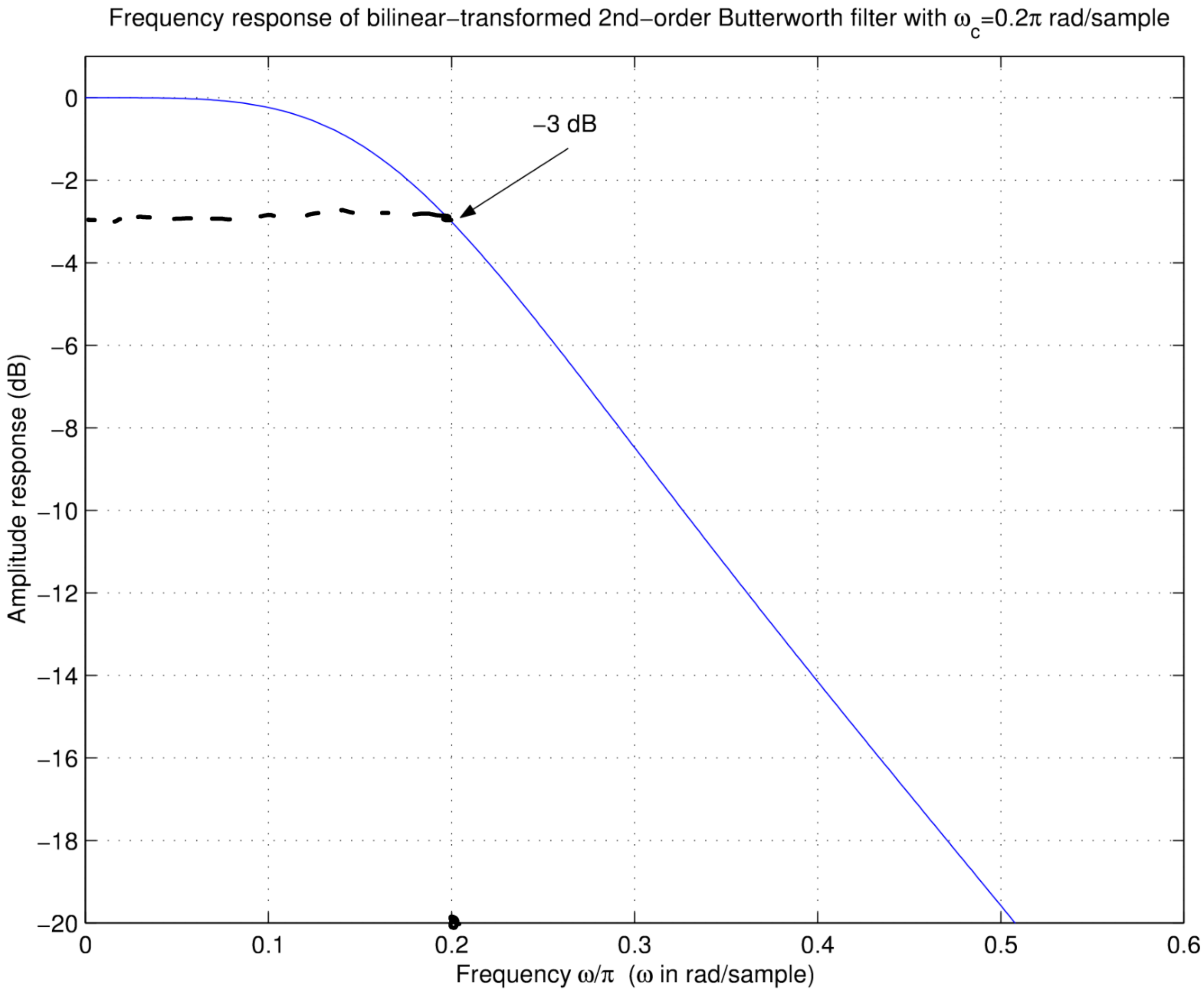
$K = 1$

$$= \frac{\Omega_c^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \sqrt{2}\,\Omega_c \frac{1-z^{-1}}{1+z^{-1}} + \Omega_c^2}$$

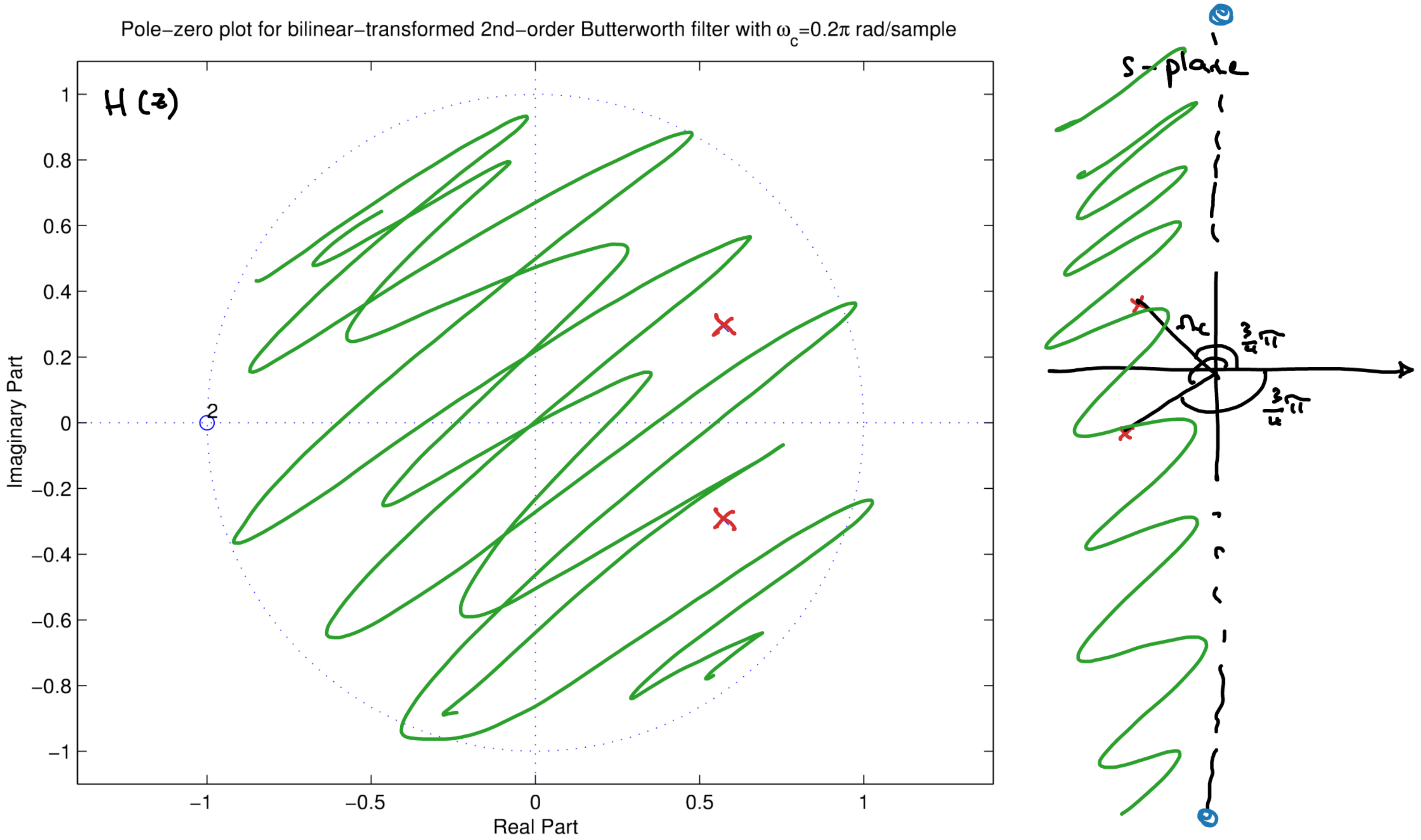
$\Omega_c = 0.325$

$$\vdots$$

$$= \frac{0.06773 + 0.135z^{-1} + 0.0677z^{-2}}{1 - 1.143z^{-1} + 0.413z^{-2}}$$



Frequency response of bilinear−transformed 2nd−order Butterworth filter with ω_c=0.2π rad/sample
−3 dB
Amplitude response (dB)
Frequency ω/π (ω in rad/sample)
0
−2
−4
−6
−8
−10
−12
−14
−16
−18
−20
0
0.1
0.2
0.3
0.4
0.5
0.6


Pole−zero plot for bilinear−transformed 2nd−order Butterworth filter with ωc=0.2π rad/sample
H(z)
Imaginary Part
Real Part
2
s-plane
Ωc
3/4 π
3/4 π