Introduction to discrete-time filters

Ideal and elementary filters

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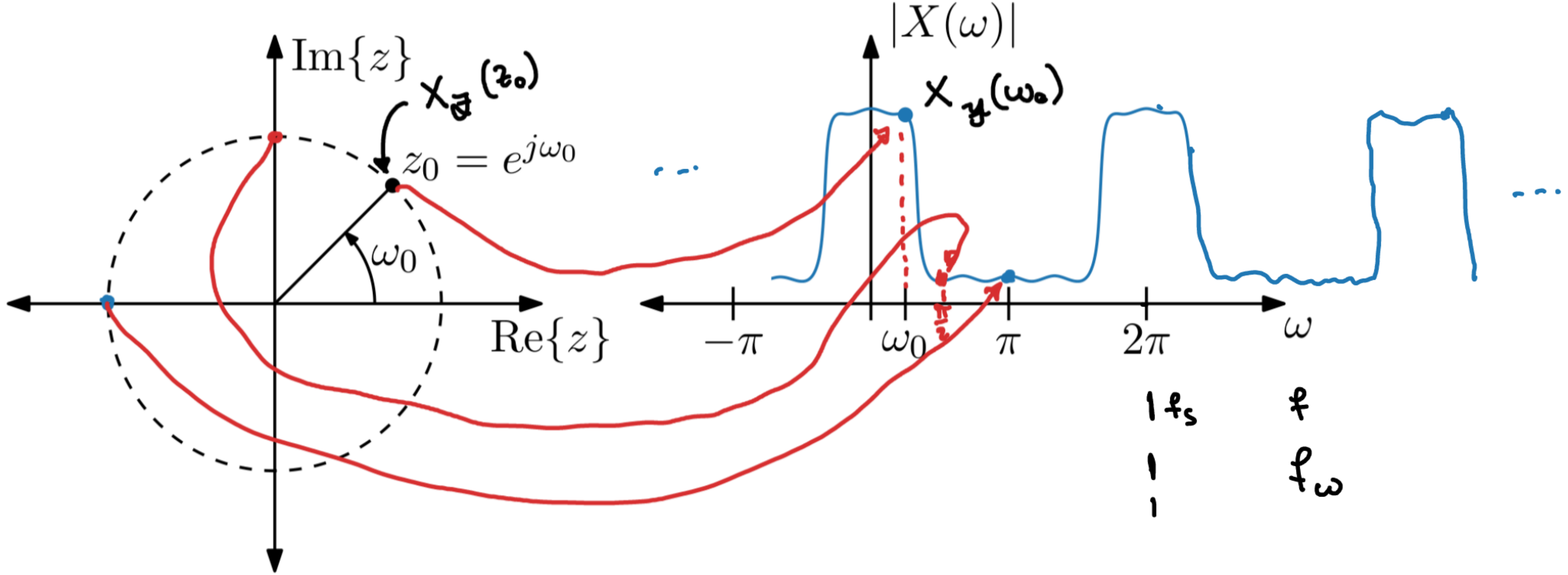
Recap: z-transform and DTFT

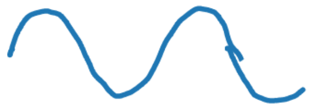
DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$X_y(\omega) = X_z(z)\Big|_{z=e^{j\omega}} = X_z(e^{j\omega})$$



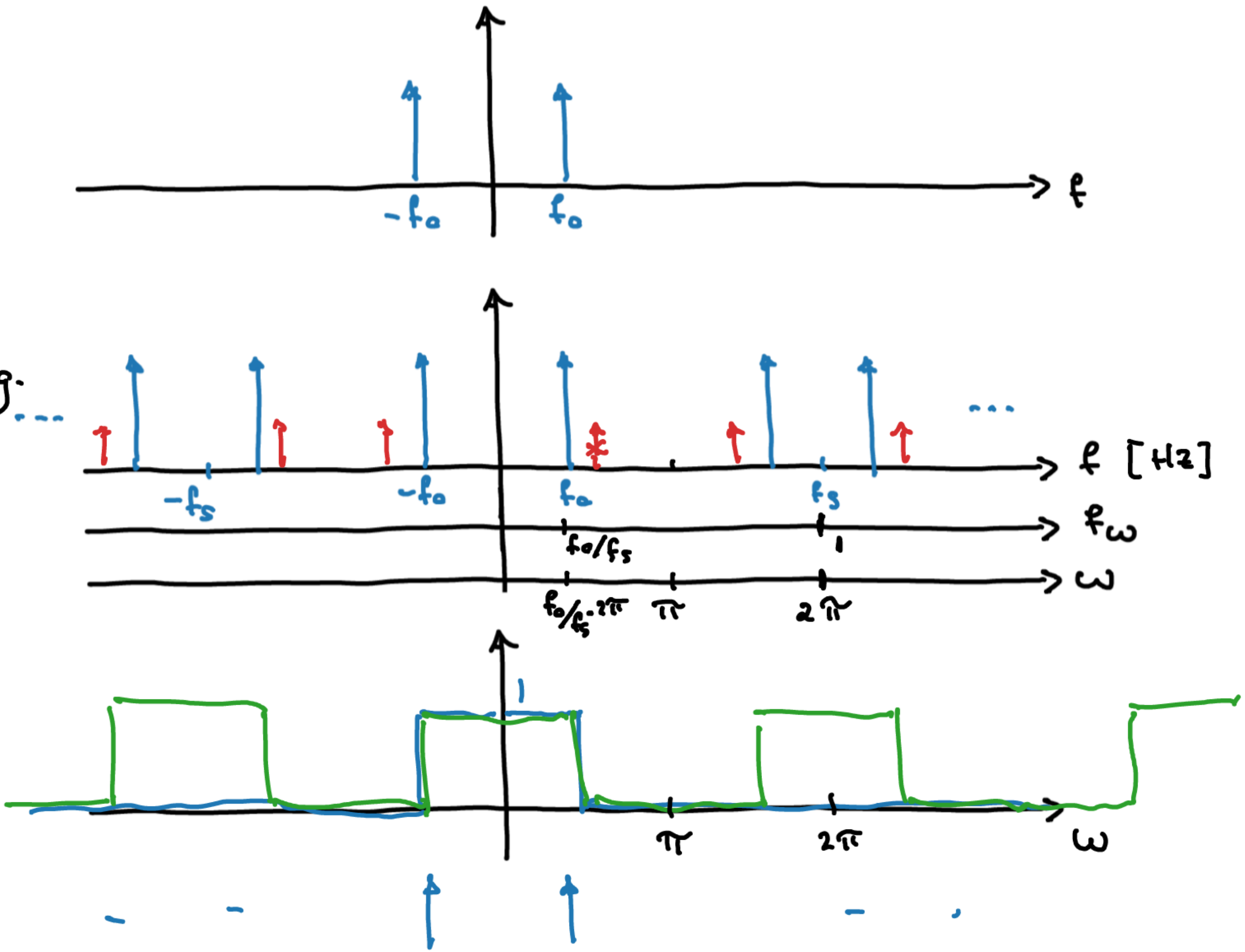
Continuous cosine:

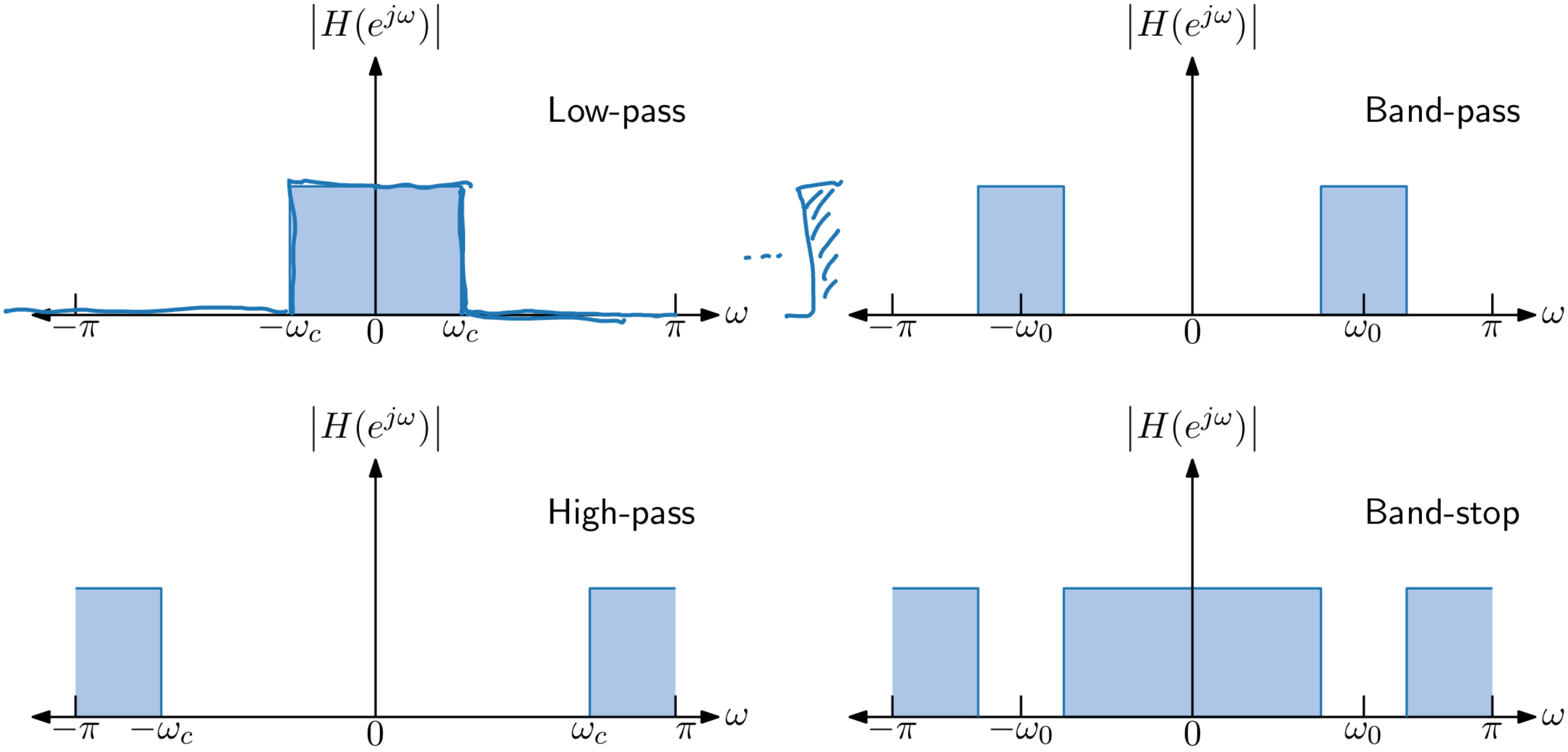
$x(t) = \cos(2\pi f_0 t)$



Sample $x(t)$ to get $x[n]$ with f_s sampling freq.

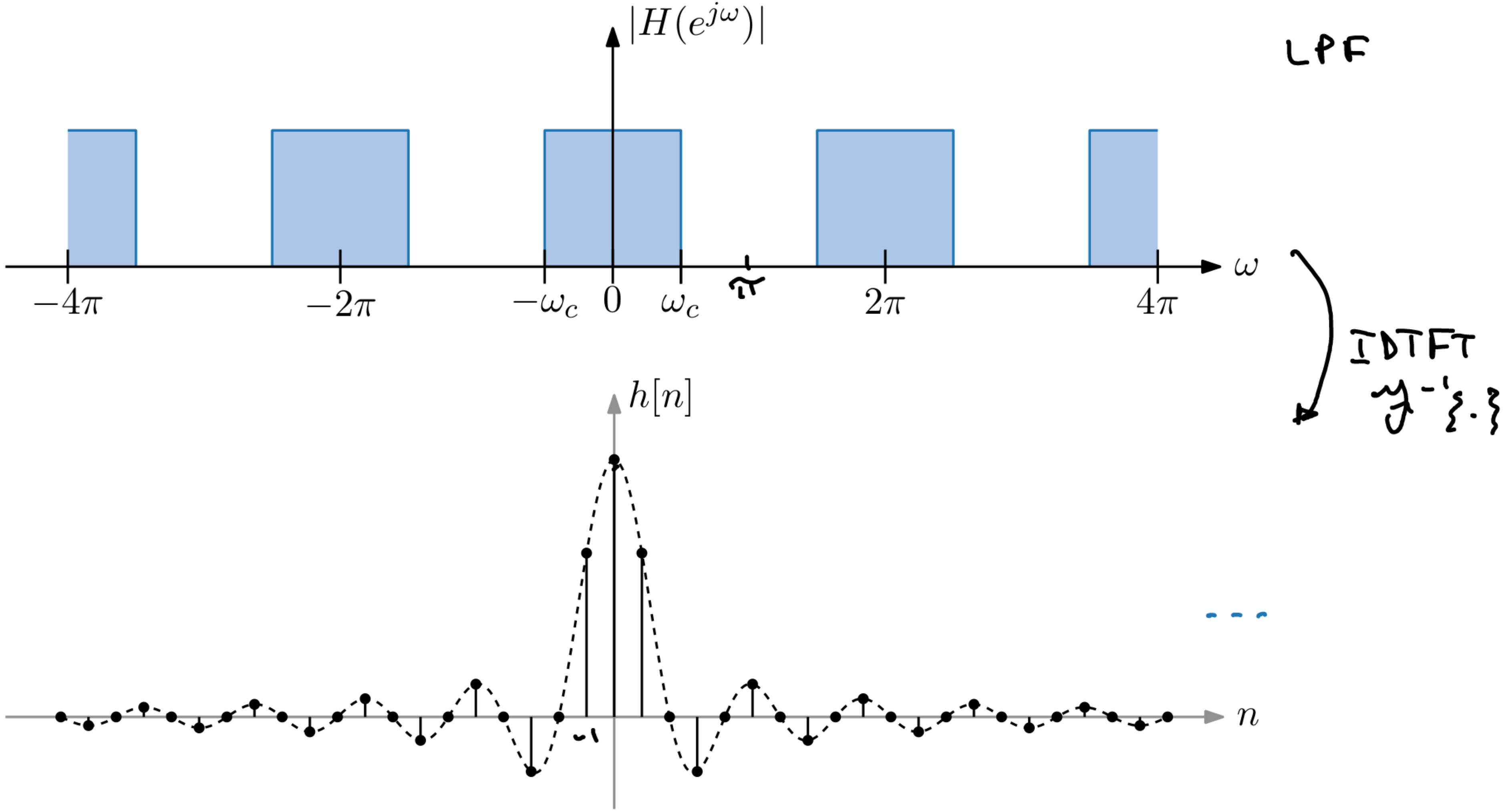
Design filter to let everything below $f_{w_0} = \frac{f_0}{f_s}$ in (LPF)





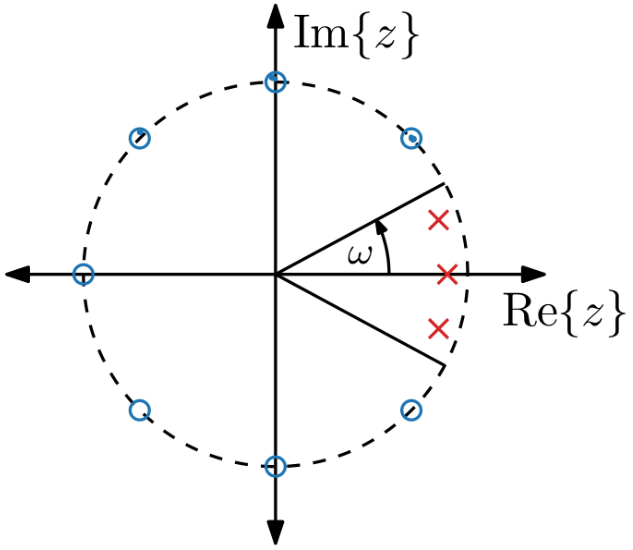
$|H(e^{j\omega})|$
Low-pass
$-\pi$ $-\omega_c$ 0 ω_c π ω
$|H(e^{j\omega})|$
Band-pass
$-\pi$ $-\omega_0$ 0 ω_0 π ω
$|H(e^{j\omega})|$
High-pass
$-\pi$ $-\omega_c$ 0 ω_c π ω
$|H(e^{j\omega})|$
Band-stop
$-\pi$ $-\omega_0$ 0 ω_0 π ω

Ideal filters are unrealisable

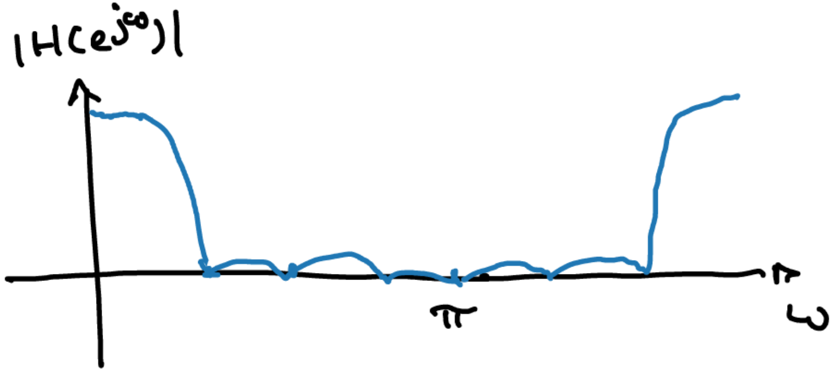


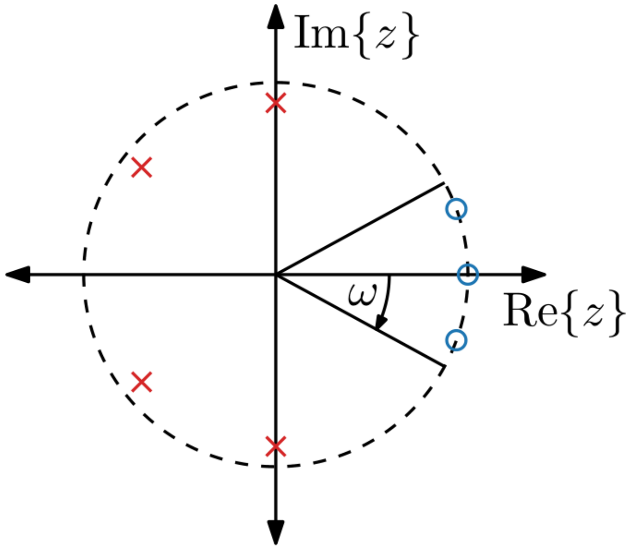
Elementary filters

LPF, BSF, HPF, BPF
(1) (2) (3) (4)



LPF

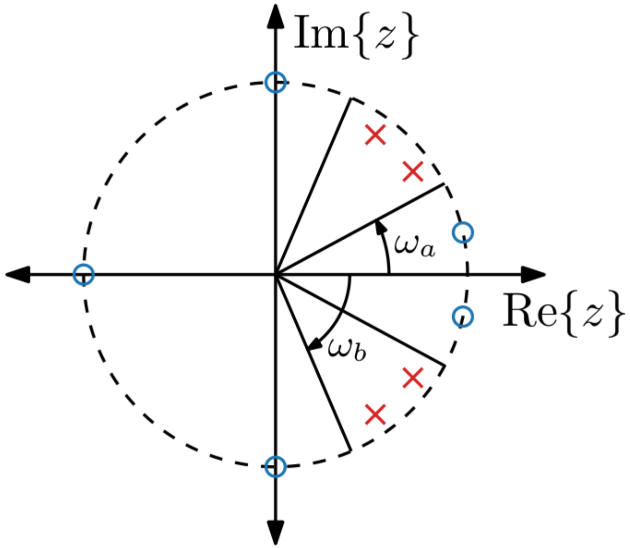


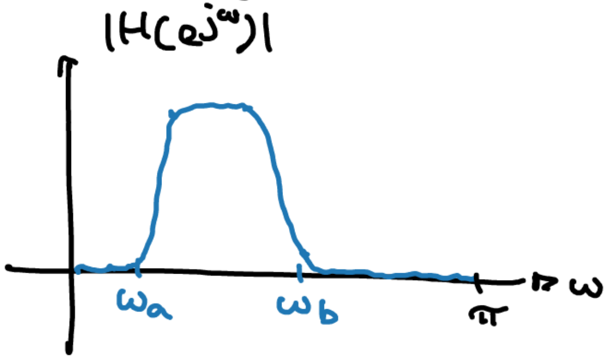


HPF

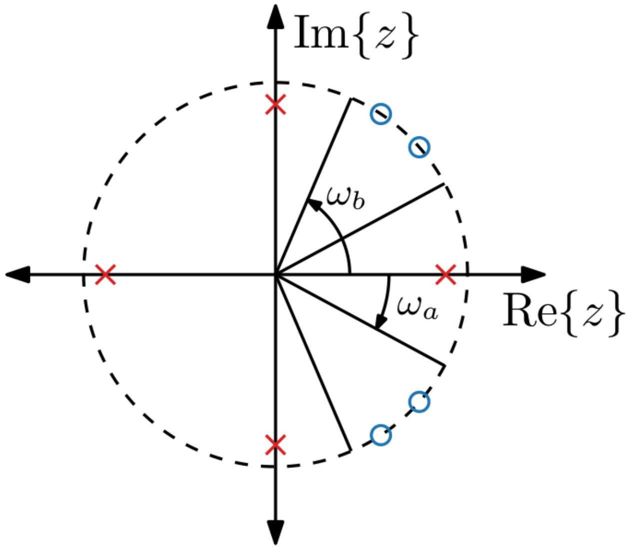
Elementary filters

LPF, BSF, HPF, BPF
(1) (2) (3) (4)





BPF



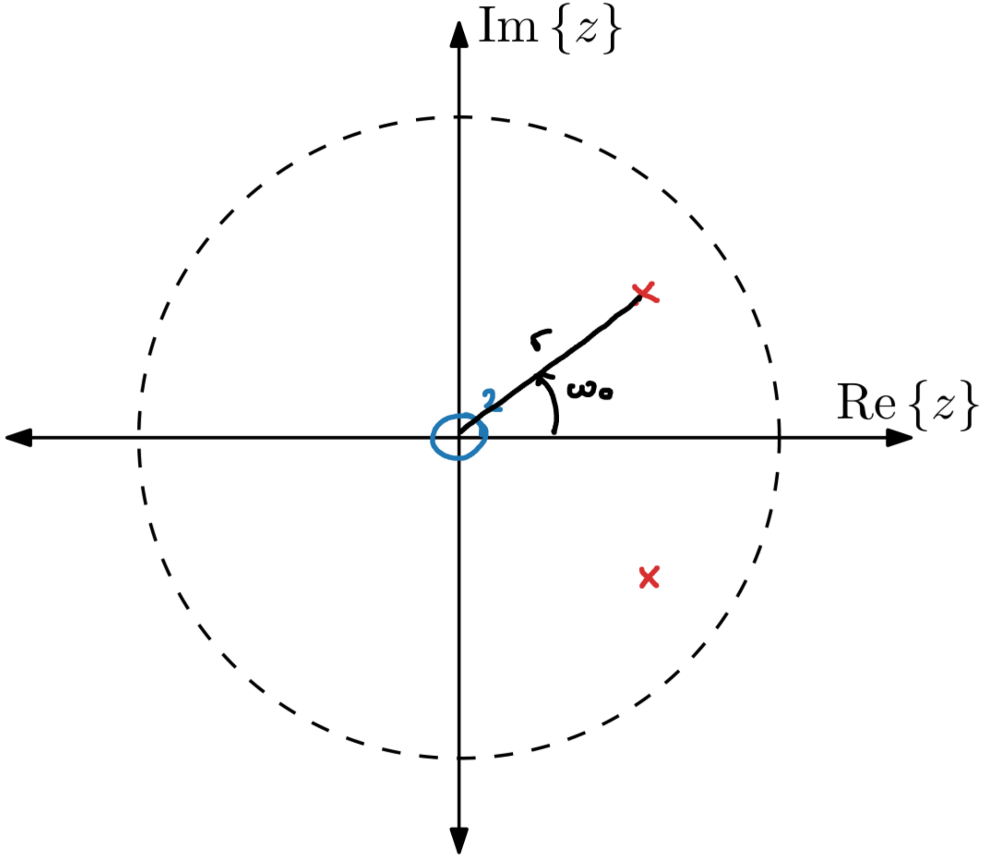
BSF

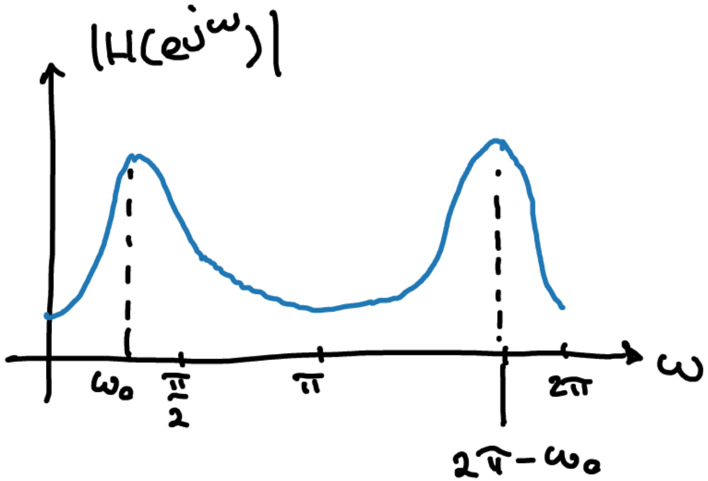
Digital resonator: An elementary BPF

$$H(z) = \frac{b_0}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} = \frac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

$$\frac{re^{j\omega_0}}{z}$$

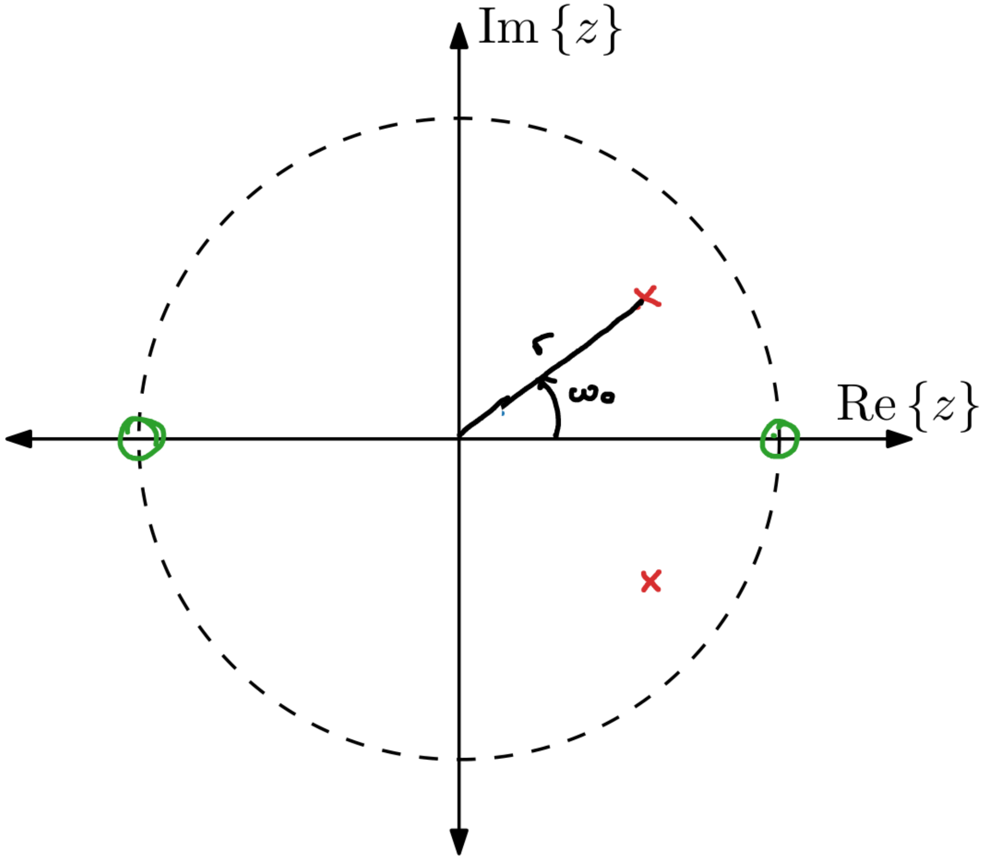
$$p_{1,2} = re^{\pm j\omega_0}$$

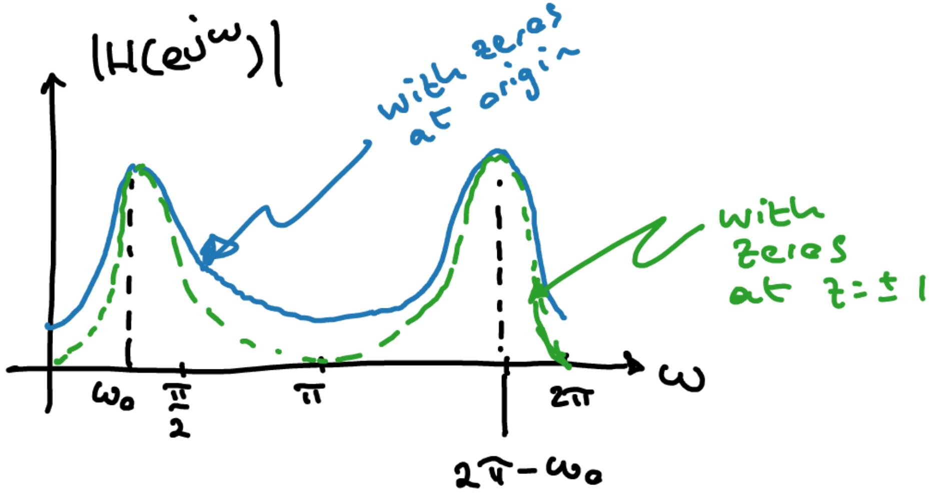


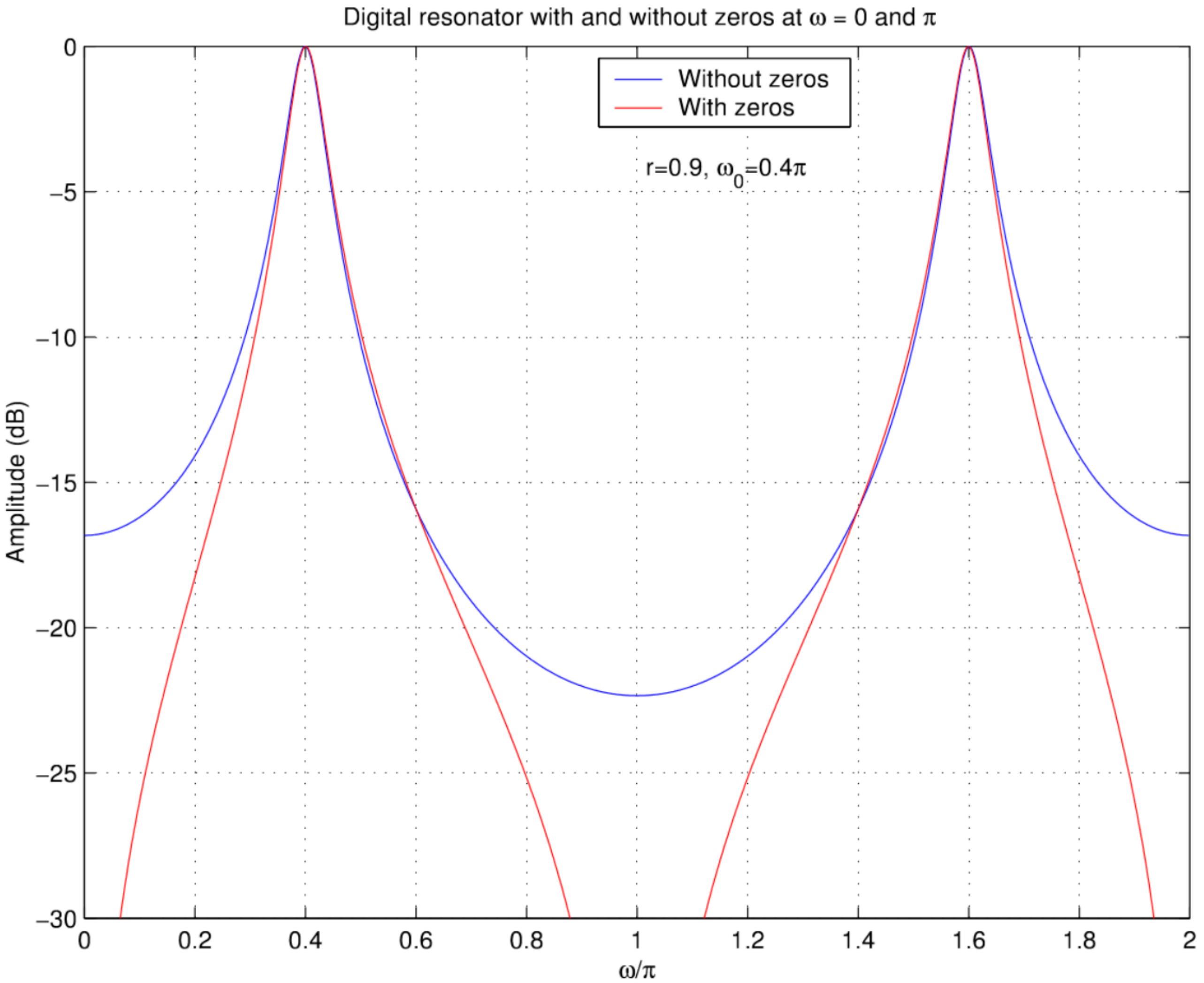


Can deepen nulls by introducing zeros at $z = \pm 1$:

$$H(z) = \frac{b_0(1 - z^{-2})}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}} = \frac{b_0(1 - z^{-1})(1 + z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$



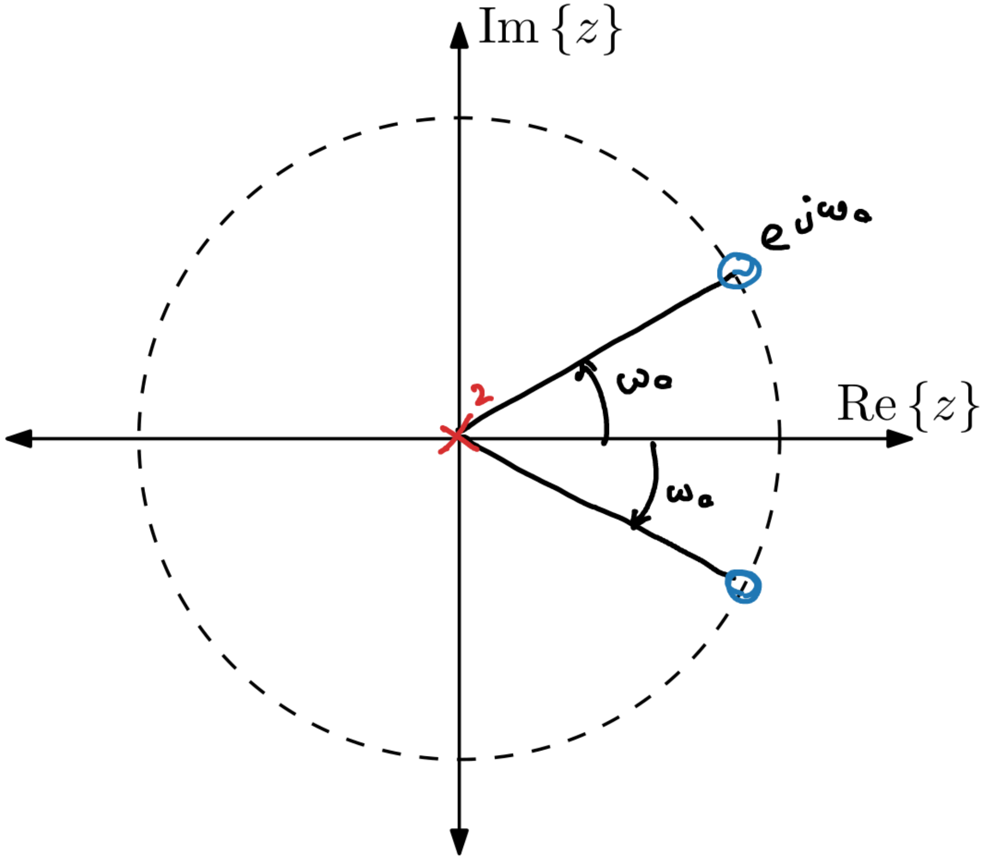


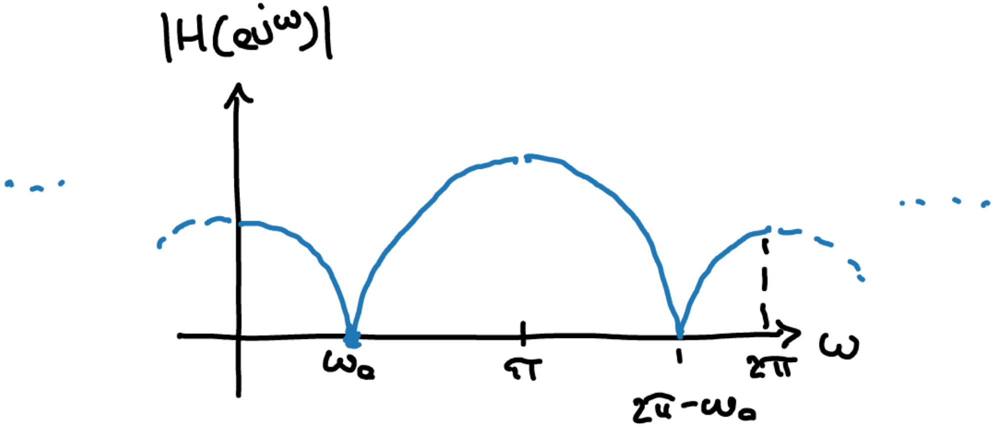

Digital resonator with and without zeros at ω = 0 and π
Without zeros
With zeros
r=0.9, ω0=0.4π
Amplitude (dB)
ω/π
0
−5
−10
−15
−20
−25
−30
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Notch filter: An elementary BSF

$$H(z) = b_0(1 - (2\cos\omega_0)z^{-1} + z^{-2}) = b_0\left(\underbrace{1 - e^{j\omega_0}z^{-1}}_{1 - e^{j\omega_0}/z}\right)\left(1 - e^{-j\omega_0}z^{-1}\right)$$

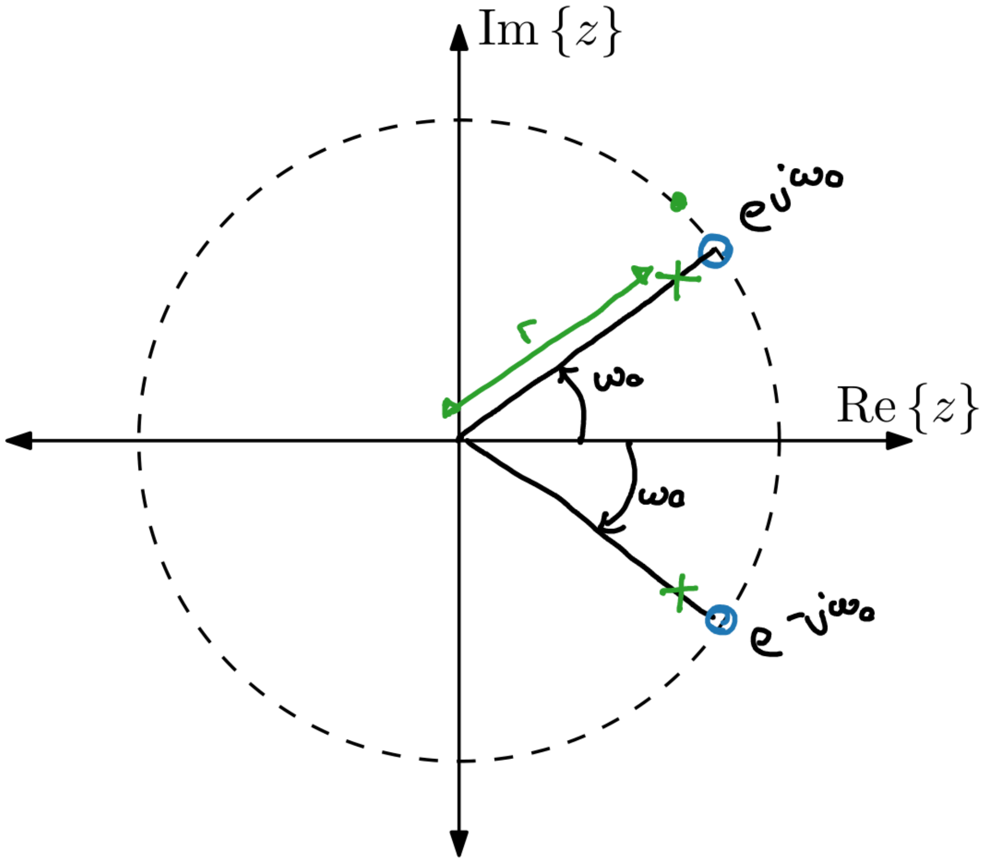
zeros: $e^{\pm j\omega_0}$

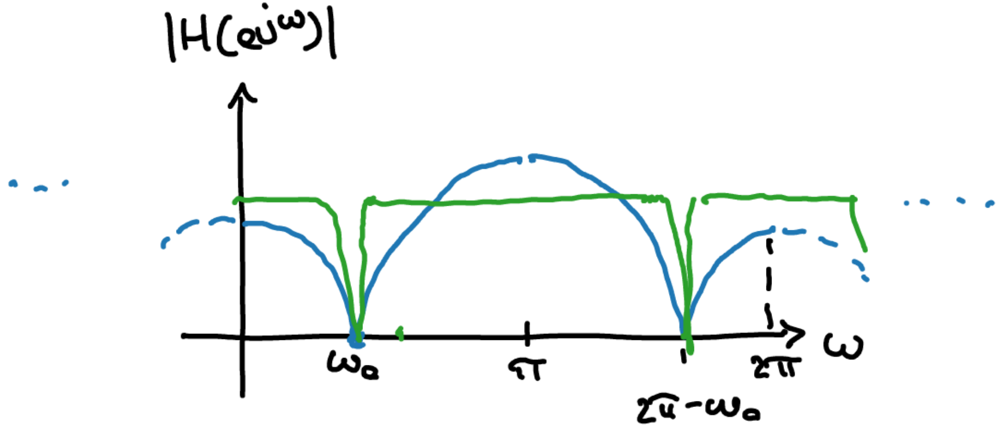


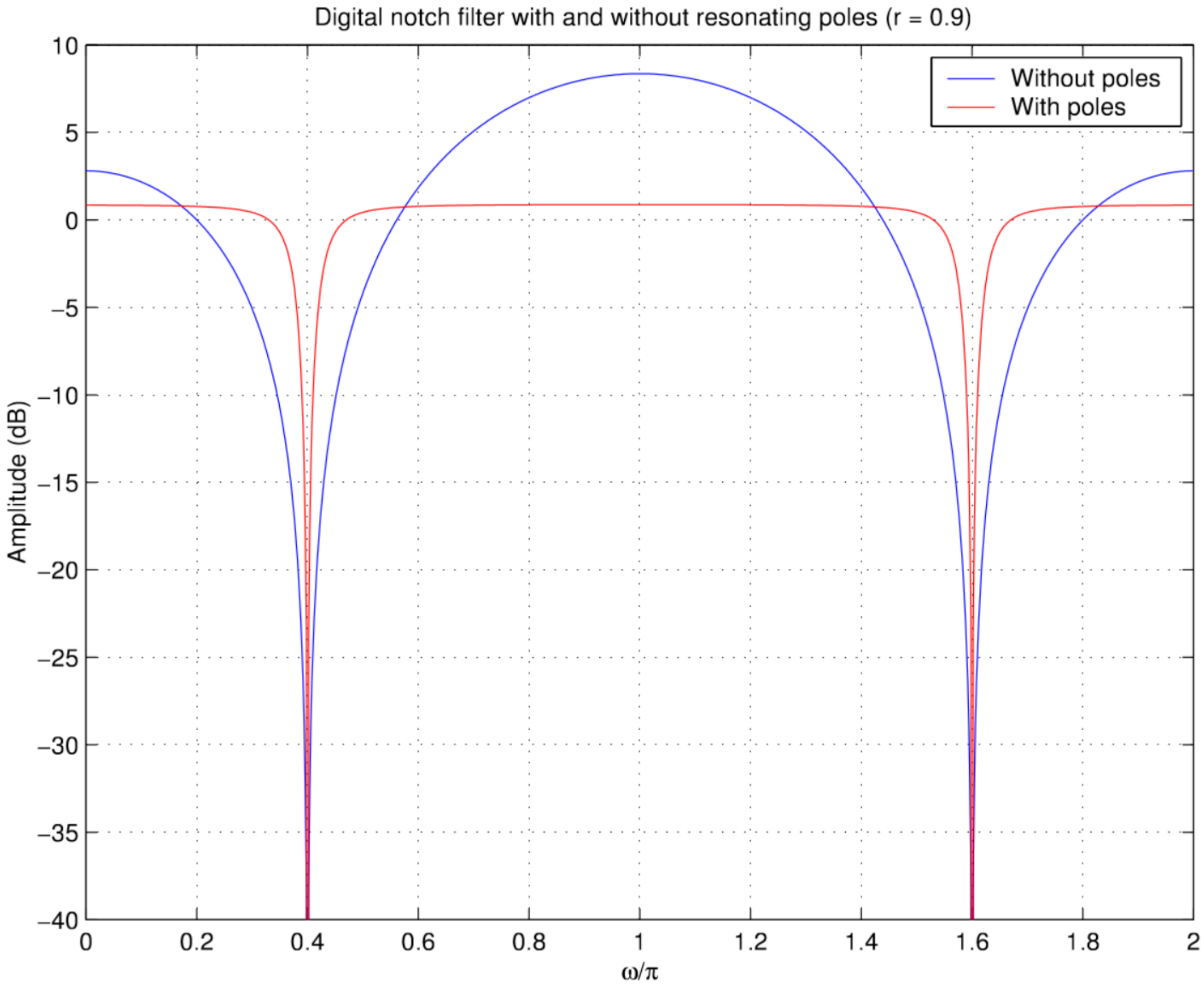


Bandwidth of notches can be reduced by placing a pole at the same frequency close to the unit circle:

$$H(z) = b_0 \frac{1 - (2\cos\omega_0)z^{-1} + z^{-2}}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}} = b_0 \frac{\left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)}{\left(1 - re^{j\omega_0} z^{-1}\right)\left(1 - re^{-j\omega_0} z^{-1}\right)}$$

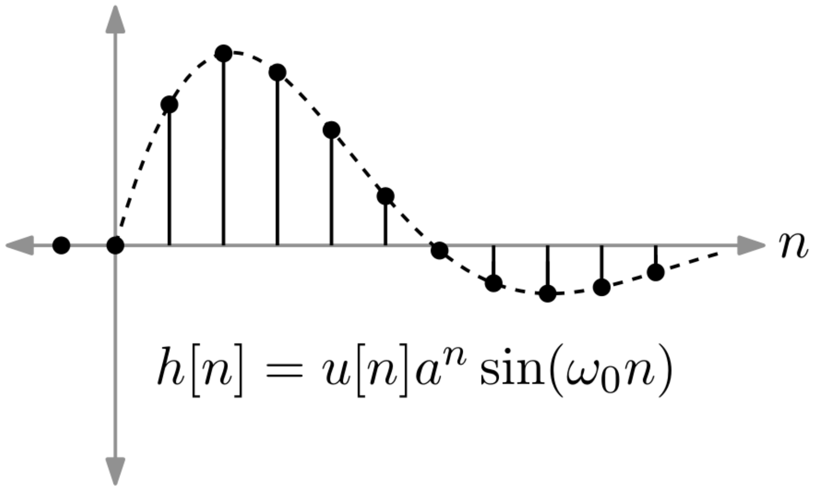





Digital notch filter with and without resonating poles (r = 0.9)
Without poles
With poles
Amplitude (dB)
ω/π
10
5
0
−5
−10
−15
−20
−25
−30
−35
−40
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Why not put poles as close as possible to unit circle?

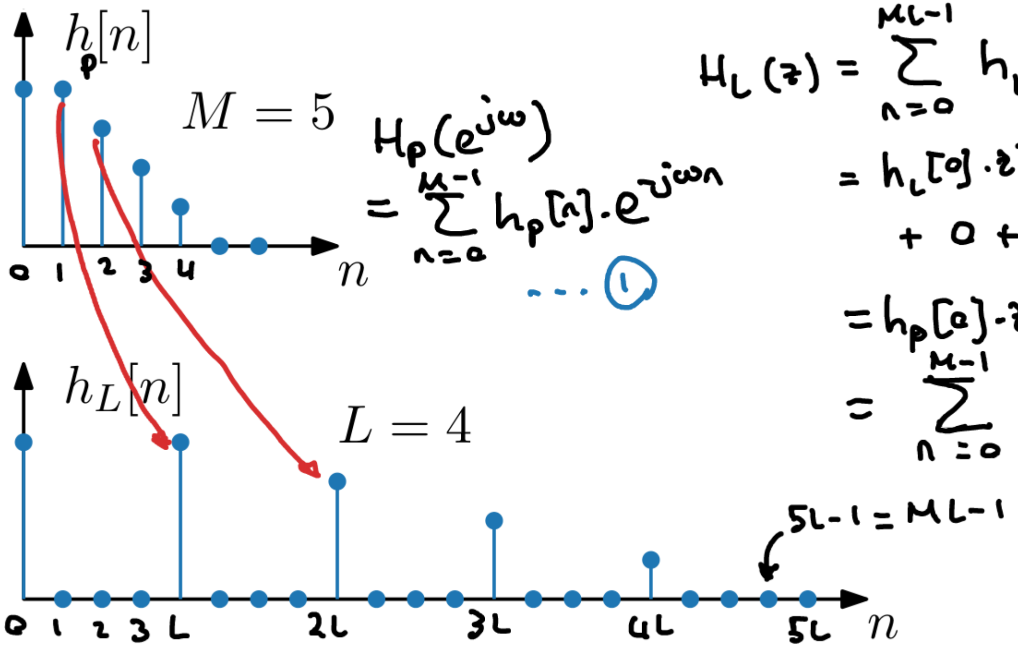
- The derivation that $y[n] = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\omega n + \angle H(e^{j\omega})}$ is for steady state
- This does not take transient effects into account
- What happens below when a gets close to 1? (similar for damped cosine)



$$H(z) = \frac{(a \sin \omega_0) z^{-1}}{1 - (2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$$
$$= \frac{(a \sin \omega_0) z^{-1}}{\left(1 - a e^{j\omega_0} z^{-1}\right)\left(1 - a e^{-j\omega_0} z^{-1}\right)}$$

$$|z| > |a|$$

Comb filter



$$H_p(e^{j\omega}) = \sum_{n=0}^{M-1} h_p[n]\cdot e^{-j\omega n} \quad \dots (1)$$

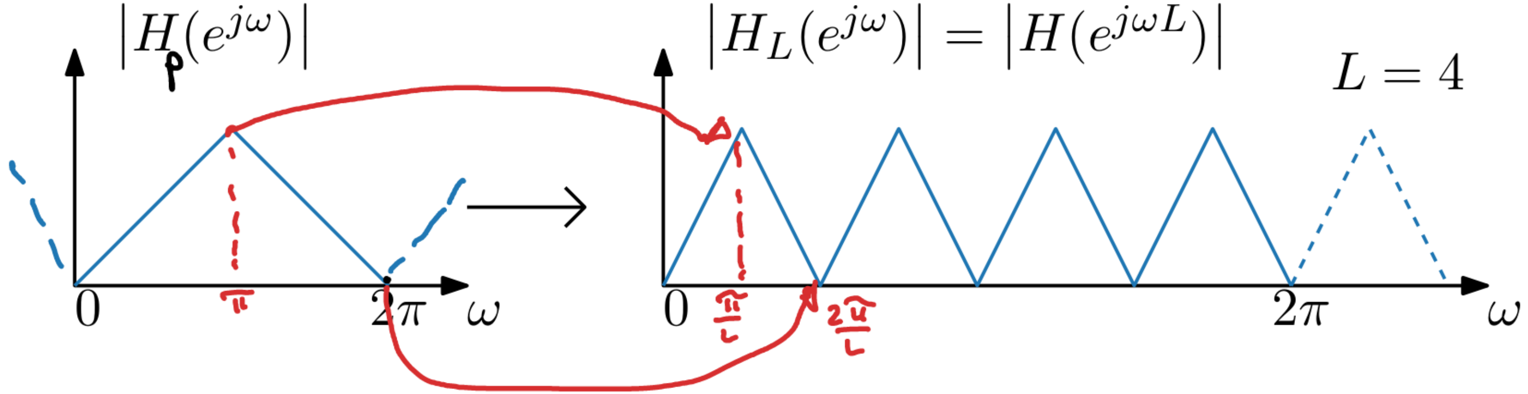
$$H_L(z) = \sum_{n=0}^{ML-1} h_L[n]\cdot z^{-n}$$

$$= h_L[0]\cdot z^{-0} + \cancel{h_L[1]}\, z^{-1} + \cancel{h_L[2]}\, z^{-2} + \cancel{h_L[3]}\, z^{-3} + h_L[L]\cdot z^{-L} + 0 + \dots + h_L[2L]\cdot z^{-2L} + \dots h_L[3L]\cdot z^{-3L} + \dots + h_L[4L]\cdot z^{-4L}$$

$$= h_p[0]\cdot z^{-0} + h_p[1]\, z^{-L} + h_p[2]\, z^{-2L} + h_p[3]\, z^{-3L} + h_p[L]\cdot z^{-4L}$$

$$= \sum_{n=0}^{M-1} h_p[n]\cdot z^{-nL}$$

$$H_L(e^{j\omega}) = \sum_{n=0}^{M-1} h_p[n]\cdot e^{-j\omega nL} = H_p(e^{j\omega L}) \quad (1)$$

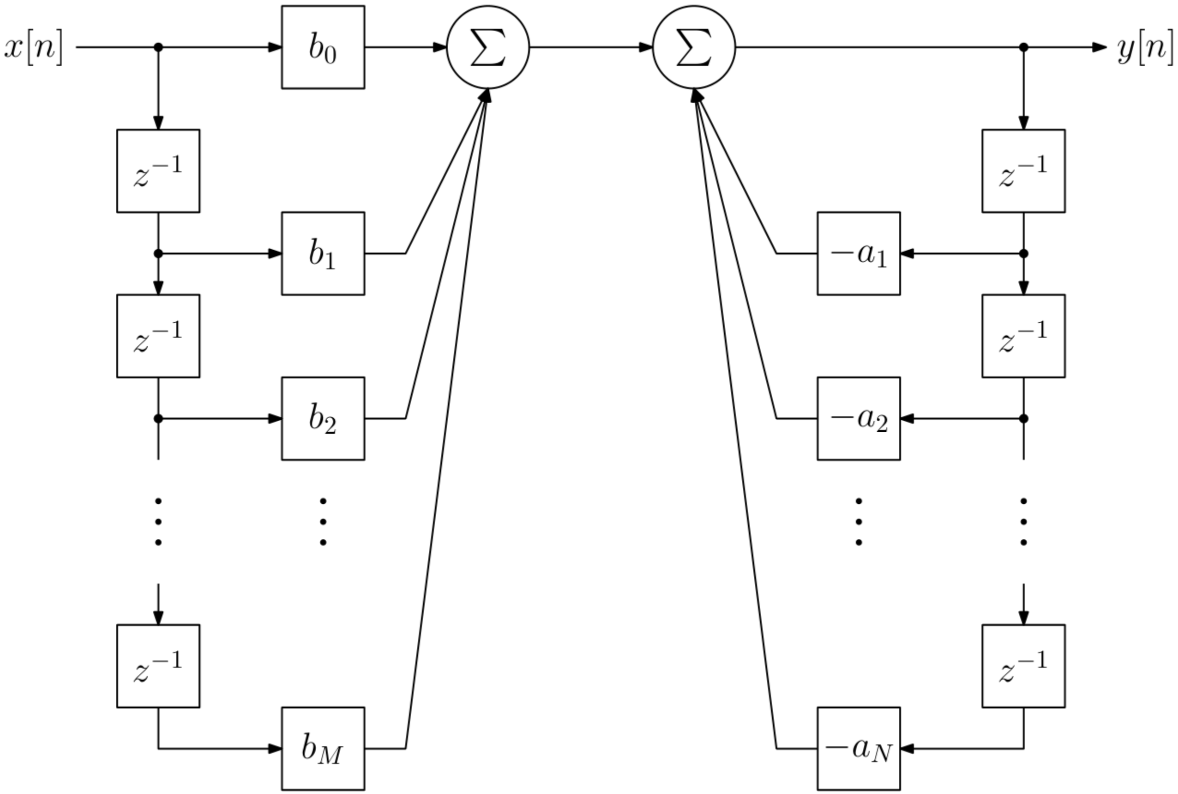


$x(t)$

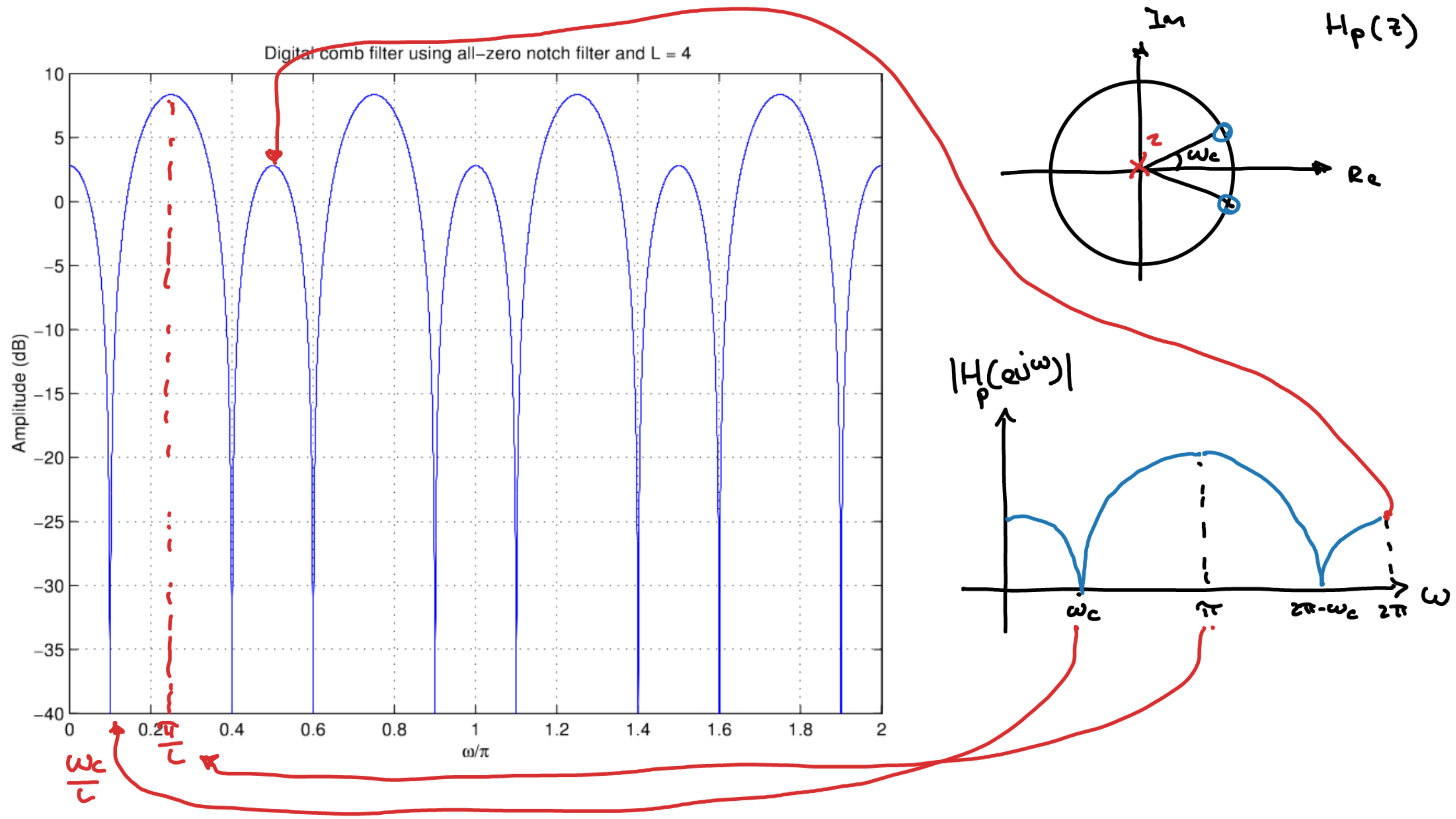
$y(t) = x(at)$

Comb filter

$$\underline{b}_p = [1\ ,\ 0\ ,\ -1] \xrightarrow{L=4} \underline{b}_L = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1]$$



$$z^{-1} \rightarrow z^{-L}$$



Digital comb filter using all−zero notch filter and L = 4
Amplitude (dB)
ω/π
$\frac{\omega_c}{L}$
$\frac{\pi}{L}$
Im
Re
$H_p(z)$
ω_c
$|H_p(e^{j\omega})|$
ω_c
π
$2\pi-\omega_c$
2π
ω

Why don't we just use the FFT to filter?

- Take the FFT of a signal
- Zero out the components we do not want
- Take the IFFT