LTI systems with the z-transform

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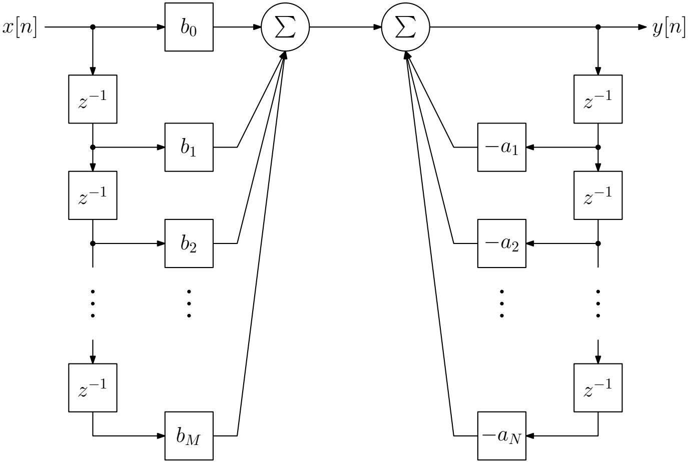


- What do you call a LCCDE system where $N = 0$? FIR
- What can you tell me about the impulse response of system where $N > 0$? IIR
- When is an LTI system BIBO stable? $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Three identities we will use below

- Linearity:

$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha \mathcal{Z}\{x[n]\} + \beta \mathcal{Z}\{y[n]\}$$

- Time shift:

$$\mathcal{Z}\{x[n-k]\} = z^{-k}\mathcal{Z}\{x[n]\}$$

- Convolution:

$$\mathcal{Z}\{x[n] * y[n]\} = \mathcal{Z}\{x[n]\} \cdot \mathcal{Z}\{y[n]\}$$

Transfer function

Linear time-invariant (LTI) system:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad y[n] = x[n] * h[n]$$

$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\{x[n]\} \cdot \mathcal{Z}\{h[n]\}$$

$$Y(z) = X(z) \cdot H(z)$$

Transfer function
$H(z) = \mathcal{Z}\{h[n]\}$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

Transfer functions of LCCDE systems

$$Y(z) = X(z) \cdot H(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)}$$

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$\Big\downarrow \ \mathcal{Z}\{\cdot\}$$

$$Y(z) = -\sum_{k=1}^{N} a_k \mathcal{Z}\{y[n-k]\} + \sum_{k=0}^{M} b_k \mathcal{Z}\{x[n-k]\}$$

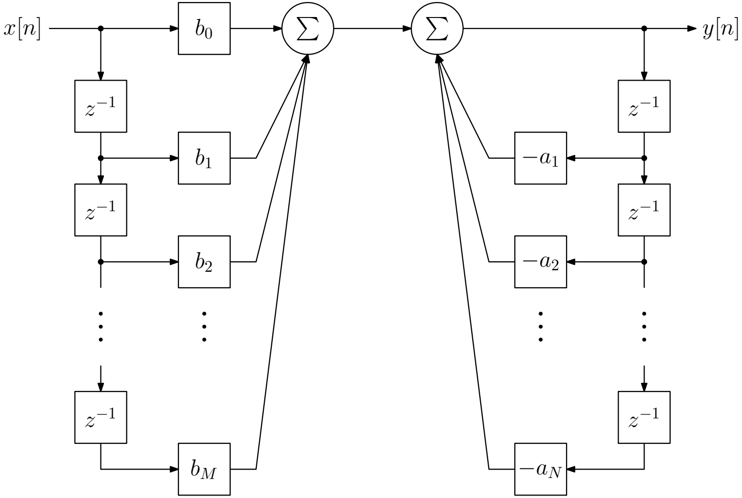
$$= -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z) \cdot \left[\sum_{k=0}^{M} b_k z^{-k}\right]$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} \; \overbrace{\times \frac{z^N}{z^N} \quad z^{N-M}}^{=1}$$

$$= \frac{b_0 z^M + b_1 z^{M-1} + \dots + b_M}{z^N + a_1 z^{N-1} + \dots + a_N} \cdot z^{N-M}$$

$$H(z) = z^{N-M}\,\frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}}$$



Poles and zeros

$$H(z) = z^{N-M} \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}} = b_0 z^{N-M} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$= b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Zeros: where $H(z) = 0$

Poles: where $H(z) \to \infty$

Example:

$$H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Zero: $z = 0$

Pole: $z = a$

All-pole and all-zero systems

LCCDE: $$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

With $N = 0$ we have an all-zero system $\Rightarrow$ FIR:

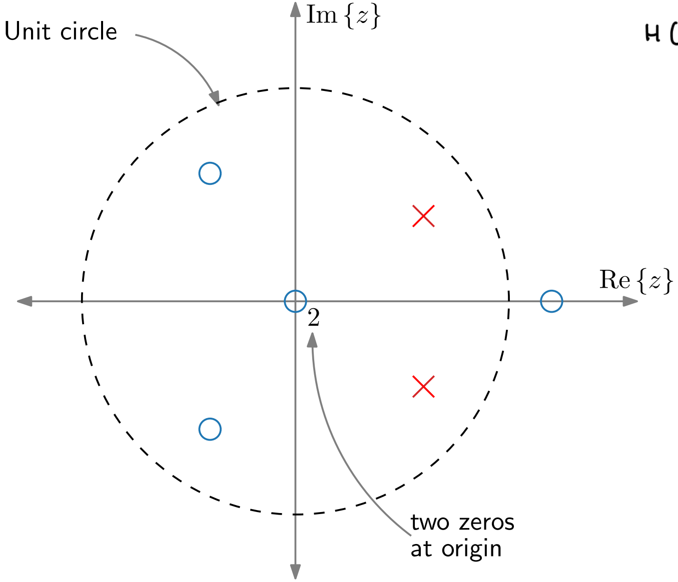
$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}$$

(Still have M poles at origin.)

With $M = 0$ we have an all-pole system $\Rightarrow$ IIR:

$$\begin{aligned} H(z) &= \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} \\ &= \frac{b_0 z^N}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \ldots + a_N} \end{aligned}$$

(Still have N zeros at origin.)



$$H(z) = A \frac{z^2 (z - z_1)(z - z_1^*)(z - z_2)}{(z - p_1)(z - p_1^*)}$$

Stability of causal LTI systems

BIBO stable: $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (necessary and sufficient)

Let us look at how the pole placement in $H(z)$ affect the summability of the resulting impulse response $h[n]$ for a few systems.

System with real poles

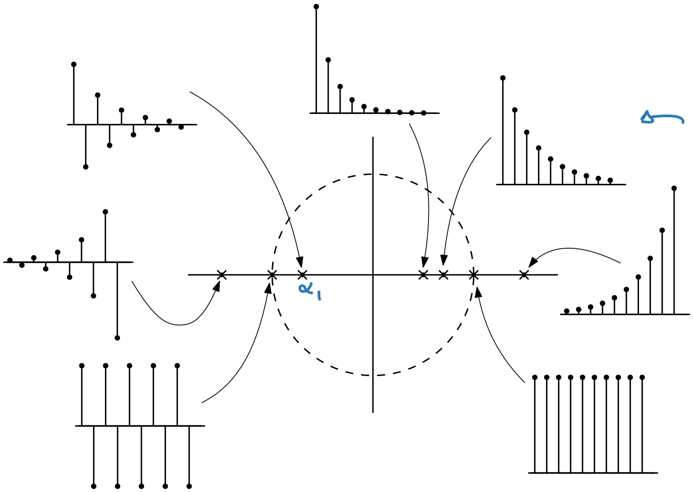
Partial fraction expansion

$$H(z) = \frac{z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{A_1}{1 - \alpha_1 z^{-1}} + \frac{A_2}{1 - \alpha_2 z^{-1}}$$

Tables

$$\therefore \quad h[n] = \left[A_1(\alpha_1)^n + A_2(\alpha_2)^n\right] u[n]$$

System with single real pole: $\quad H(z) = \dfrac{1}{1-\alpha z^{-1}} \quad \Leftrightarrow \quad h[n] = \alpha^n \, u[n]$



BIBO:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

which of these correspond to a stable h[n]?

System with complex pole pair

$$H(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

Using tables, the inverse z-transform is:

$$h[n] = \mathcal{Z}^{-1}\{X(z)\} = A \cdot p^n \cdot u[n] + A^* \cdot (p^*)^n \cdot u[n]$$

With $p = re^{j\omega_0}$:

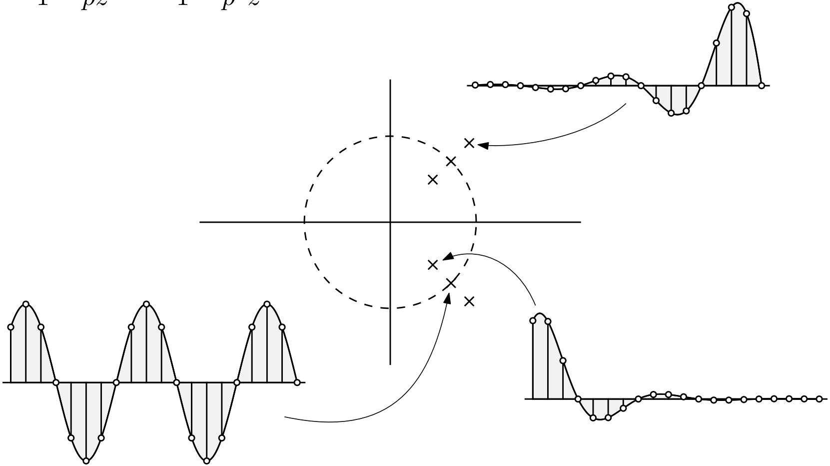
$$\begin{aligned}
h[n] &= A \cdot p^n \cdot u[n] + A^* \cdot (p^*)^n \cdot u[n] \\
&= [A \cdot p^n + A^* \cdot (p^*)^n]\; u[n] \\
&= \left[A\, r^n e^{j\omega_0 n} + A^*\, r^n e^{-j\omega_0 n}\right] u[n] \\
&= |A| r^n \left[e^{j(\omega_0 n + \angle A)} + e^{-j(\omega_0 n + \angle A)}\right] u[n] \\
&= 2|A| r^n \cos(\omega_0 n + \angle A)\, u[n]
\end{aligned}$$

$p = re^{j\omega_0}$

$A = |A| \cdot e^{j\angle A}$

System with complex pole pair, $p = re^{j\omega_0}$:

$$H(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}} \quad \Leftrightarrow \quad h[n] = 2|A| r^n \cos(\omega_0 n + \angle A)\, u[n]$$



Stability: A causal LTI system is BIBO stable if and only if its poles are inside $r < 1$.