Introduction to the z-transform

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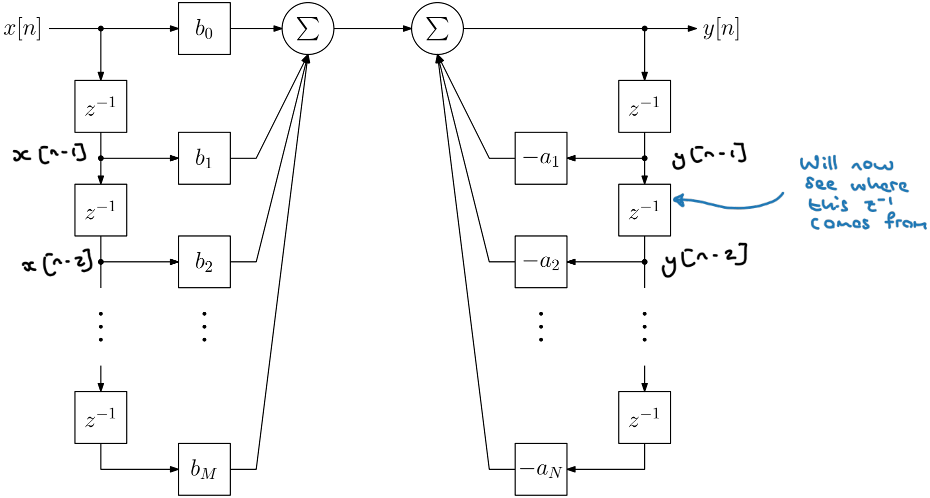
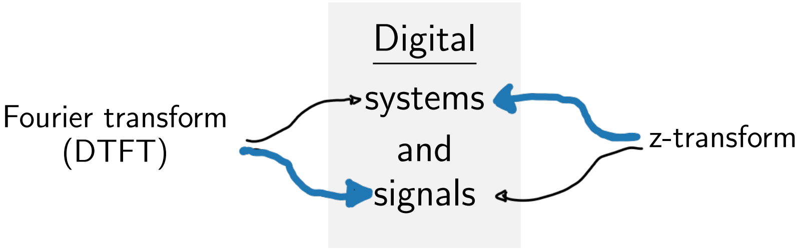
How do we know what a discrete system does to a signal?





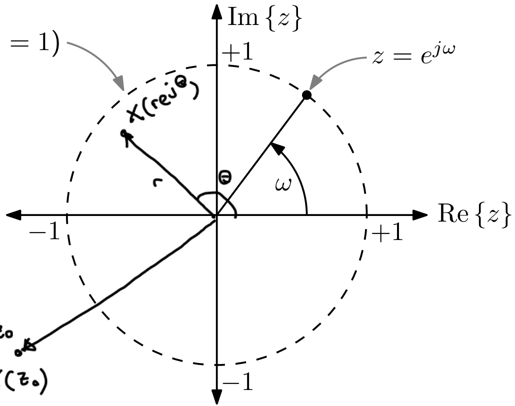
We will see that the z-transform is actually a generalised version of the DTFT, so maybe we should have a dashed connection between the two transforms on this intuitive diagram.

The z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

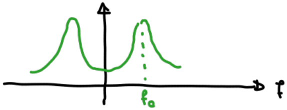
$$X(z) = \mathcal{Z}\{x[n]\}, \quad z \in \mathbb{C}$$

Unit circle
(contour with $|z| = 1$)



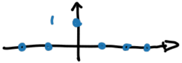
Fourier transform:

$$X(f) = \mathcal{F}\{x(t)\}$$



Simple z-transform examples

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



$x[n] = \delta[n]$ $\Longleftrightarrow$ $X(z) = \ldots + 0 + 0 + 1 + 0 + 0 + \ldots = 1$

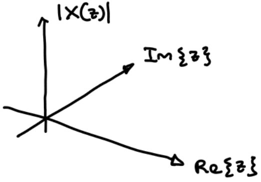
$x[0] \quad x[1] \quad x[2] \quad x[3]$

$x[n] = \{\underset{\uparrow}{1}, 2, 3, 3\}$ $\Longleftrightarrow$ $X(z) = 1 + 2z^{-1} + 3z^{-2} + 3z^{-3}$

$x[-2] \quad x[-1] \quad x[0]$

$x[n] = \{1, 2, \underset{\uparrow}{5}, 7, 0\}$ $\Longleftrightarrow$ $X(z) = 1 \cdot z^{-(-2)} + 2z^{+1} + 5 + 7z^{-1} + 0 + \ldots$

A function taking a complex number
and produces a complex number:



$X(z) = z$

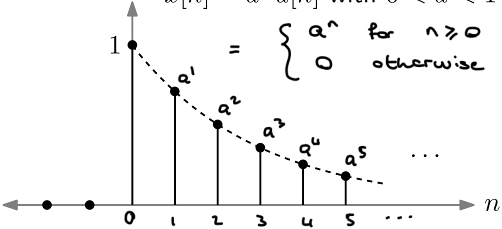
$$X(z) = \frac{1}{z - 0.58}$$

Region of convergence (ROC). The z-transform exists only for those values of z for which the infinite sum converges. For a particular signal $x[n]$, the values of z for which this is true is the region of convergence (ROC) of the z-transform $X(z)$.

Another z-transform example

$x[n] = a^n u[n]$ with $0 < a < 1$

$$= \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

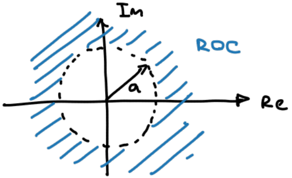
$$= \sum_{n=0}^{\infty} (\underbrace{a \cdot z^{-1}}_{r})^n$$

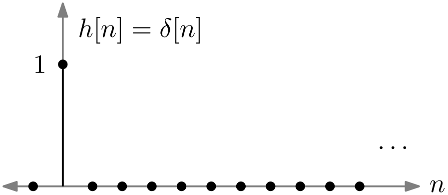
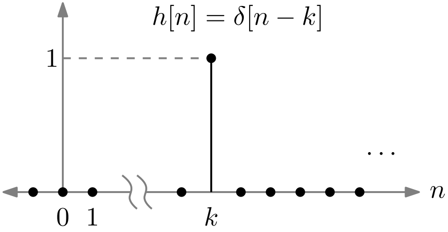
$$= \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1, \quad |a| < |z|$$

Identities:

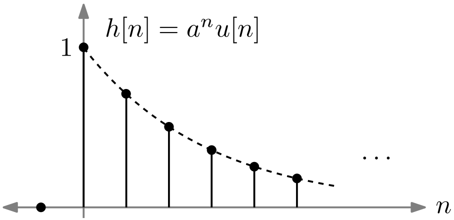
$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad \text{for } |r| < 1$$



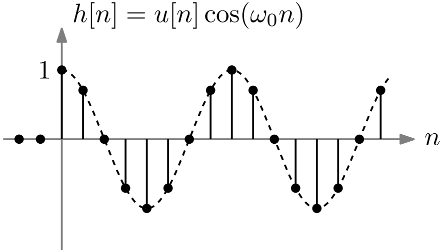
Discrete time-domain ⇔ z-transform	
	$H(z) = 1$ ROC : all z
	$H(z) = z^{-k} = \frac{1}{z^k}$ $z \neq 0$ when $k>0$ $z \neq \infty$ when $k<0$

Discrete time-domain ⇔ z-transform



$$H(z) = \frac{1}{1 - az^{-1}}$$

$$|z| > |a|$$



$$H(z) = \frac{1 - (\cos\omega_0)z^{-1}}{1 - (2\cos\omega_0)z^{-1} + z^{-2}}$$

$$|z| > 1$$

Properties of the z-transform

- Linearity:

$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha \mathcal{Z}\{x[n]\} + \beta \mathcal{Z}\{y[n]\}$$

- Time shift:

$$\mathcal{Z}\{x[n-k]\} = z^{-k} \mathcal{Z}\{x[n]\}$$

- Convolution:

$$\mathcal{Z}\{x[n] * y[n]\} = \mathcal{Z}\{x[n]\} \cdot \mathcal{Z}\{y[n]\}$$

- Initial-value theorem:

$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{z \to \infty} X(z) = x[0]$$

- Final-value theorem:

$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X(z)$$

Final-value theorem example

Theorem:

$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)X(z)$$

Example:

$$x[n] = u[n] \quad \Leftrightarrow \quad X(z) = \frac{1}{1 - z^{-1}}$$

$$\lim_{n\to\infty} x[n] = 1$$

$$\lim_{z\to 1} (z-1)\, X(z)$$

$$= \lim_{z\to 1} (z-1) \cdot \frac{1}{1 - z^{-1}} \quad \times \frac{z}{z}$$

$$= \lim_{z\to 1} \frac{(z-1)\cdot z}{(z-1)}$$

$$= \lim_{z\to 1} z = 1$$