Frequency response with the z-transform

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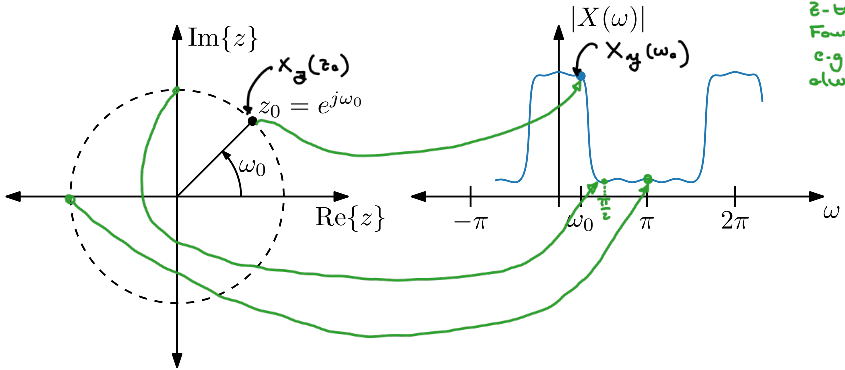
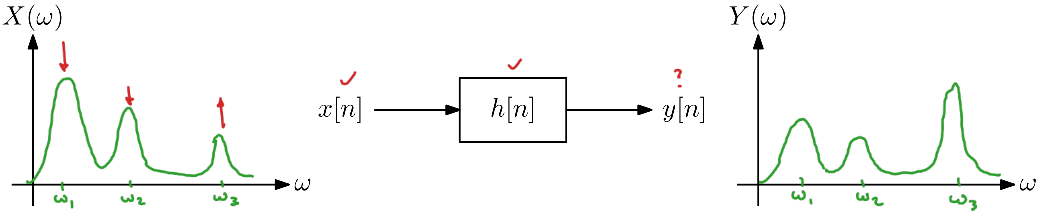
Relationship between z-transform and Fourier transform

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$X_{\mathcal{F}}(\omega) = X_{\mathcal{Z}}(z)\Big|_{z=e^{j\omega}} = X_{\mathcal{Z}}(e^{j\omega})$$

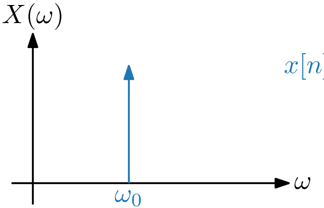
Sometimes drop subscript, but from context you will know whether z-transform or Fourier transform, e.g. $X(e^{j\omega})$ is always z-transform

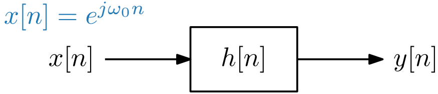


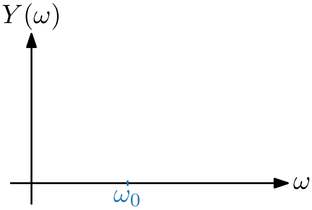


E.g. guitar chord

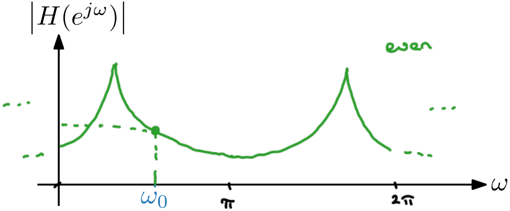
Frequency response

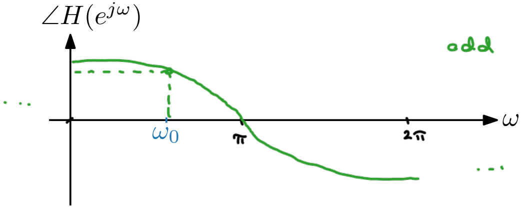






$$y[n] = H_f(\omega_0) \cdot e^{j\omega_0 n}$$
$$= H_{\mathcal{F}}(e^{j\omega_0}) \cdot e^{j\omega_0 n} = |H(e^{j\omega_0})| \cdot e^{j(\omega_0 n + \angle H(e^{j\omega_0}))}$$





Proof for frequency response

$$x[n] = e^{j\omega n}$$

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[n-i]$$

$$e^{j\omega n} \cdot e^{-j\omega i}$$

$$= \sum_{i=-\infty}^{\infty} h[i] \cdot e^{j\omega(n-i)}$$

$$= \left[\sum_{i=-\infty}^{\infty} h[i] \cdot e^{-j\omega i}\right] e^{j\omega n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$= H_y(\omega) \cdot e^{j\omega n}$$

$$= H_z(e^{j\omega}) \cdot e^{j\omega n}$$

Frequency response: z-plane interpretation

LCCDE:

$$H(z) = b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

$$\Big\downarrow{\scriptstyle z = e^{j\omega}}$$

Frequency response:

$$H(e^{j\omega}) = b_0 \underbrace{e^{j\omega(N-M)}}_{\text{Magnitude} = 1} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

$e^{j\omega k} = 1$

$\angle e^{j\omega k} = \omega k$

Magnitude response:

$$\left|H(e^{j\omega})\right| = |b_0| \frac{\prod_{k=1}^{M}|e^{j\omega} - z_k|}{\prod_{k=1}^{N}|e^{j\omega} - p_k|}$$

What happens when $e^{j\omega}$ is close to a zero? What happens to $|H(e^{j\omega})|$ when $e^{j\omega}$ is close to a pole?

Phase response:

$$\angle H(e^{j\omega}) = \angle b_0 + \omega(N-M) + \sum_{k=1}^{M}\angle(e^{j\omega} - z_k) - \sum_{k=1}^{N}\angle(e^{j\omega} - p_k)$$

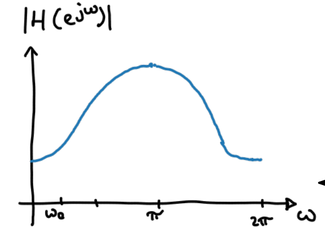
Frequency response examples

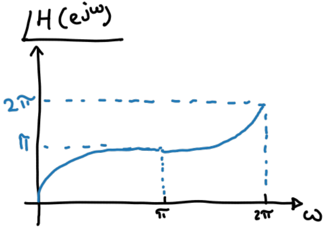
$$H(z) = z - z_1$$

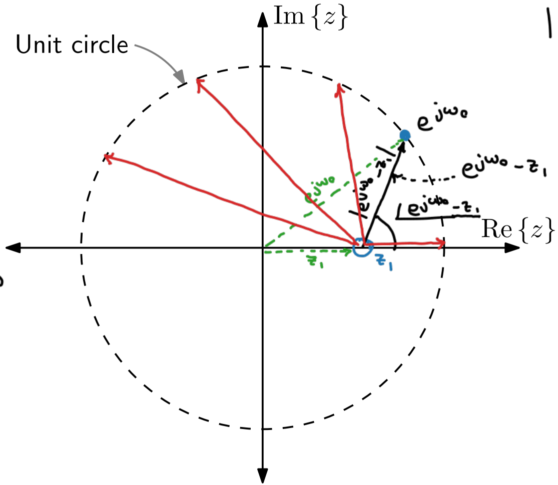
$$H(e^{j\omega}) = e^{j\omega} - z_1$$

$$|H(e^{j\omega})| = |e^{j\omega} - z_1|$$

$$\angle H(e^{j\omega}) = \angle\, e^{j\omega} - z_1$$



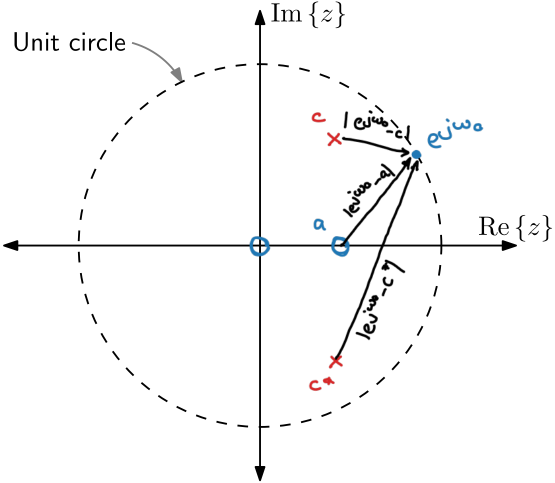




Consider:

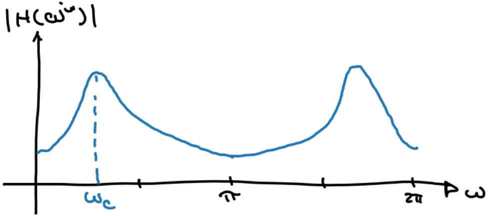
$e^{j\omega_0} - z_1 = H(e^{j\omega_0})$

$$H(z) = \frac{1 - az^{-1}}{(1 - cz^{-1})(1 - c^* z^{-1})} = \frac{z(z-a)}{(z-c)(z-c^*)}$$

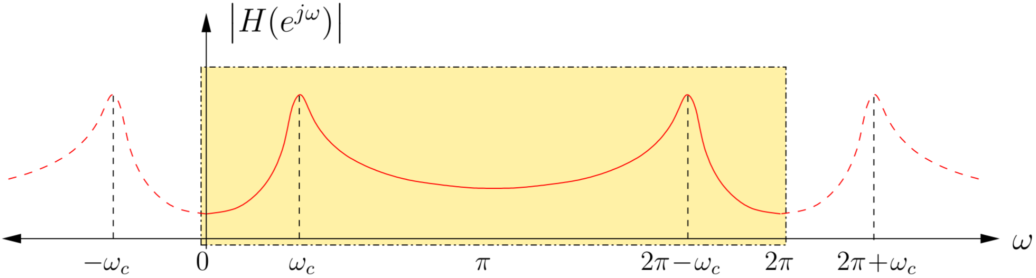


$$H(e^{j\omega}) = \frac{e^{j\omega}(e^{j\omega} - a)}{(e^{j\omega} - c)(e^{j\omega} - c^*)}$$

$$|H(e^{j\omega})| = \frac{|e^{j\omega} - a|}{|e^{j\omega} - c| \; |e^{j\omega} - c^*|}$$



In Python using scipy: freqz



$\left|H(e^{j\omega})\right|$
ω
$-\omega_c$
0
ω_c
π
$2\pi-\omega_c$
2π
$2\pi+\omega_c$