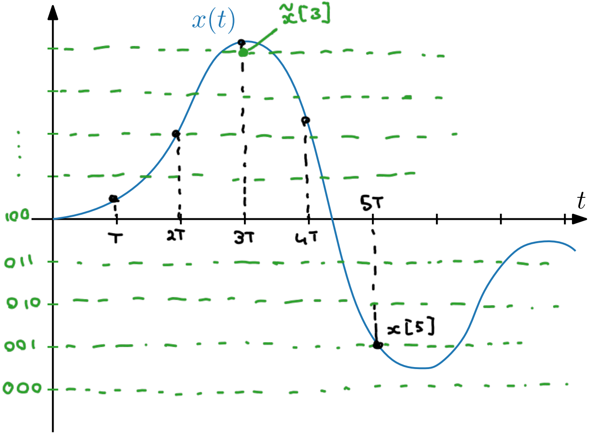
Analog-to-digital conversion

Quantisation and sampling

Herman Kamper

Analog-to-digital conversion



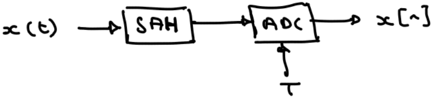
discrete

$$x[n] = x(nT)$$

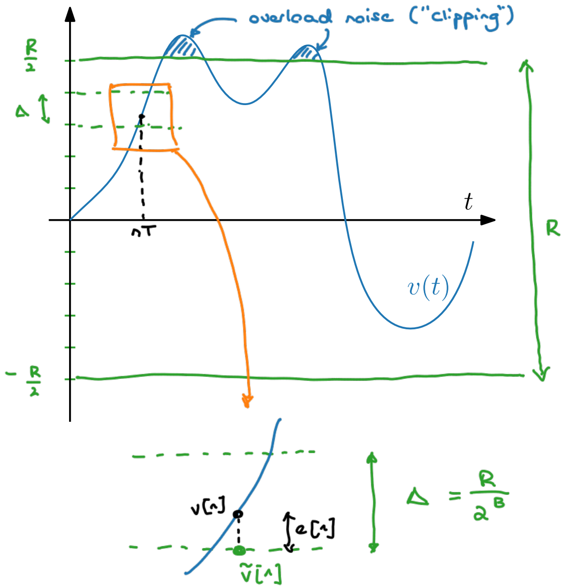
continuous

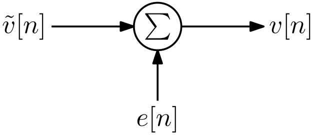
Continuous signal is discretised in two ways:

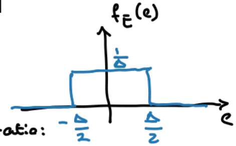
1. Discrete in time
2. Discrete in amplitude



Quantisation error analysis







Signal-to-quantisation-noise ratio:

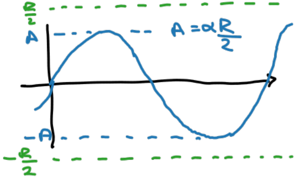
$$\mathrm{SQNR} = 10\log_{10}\frac{P_v}{P_q} \quad \text{(dB)}$$

$$P_q = \mathrm{var}[e] = \sigma_e^2 = \frac{1}{12}\left(\frac{\Delta}{2} - \left(-\frac{\Delta}{2}\right)\right)^2 = \frac{\Delta^2}{12}$$

$$= \frac{1}{12}\left(\frac{R}{2^B}\right)^2 = \frac{R^2}{12 \cdot 2^{2B}}$$

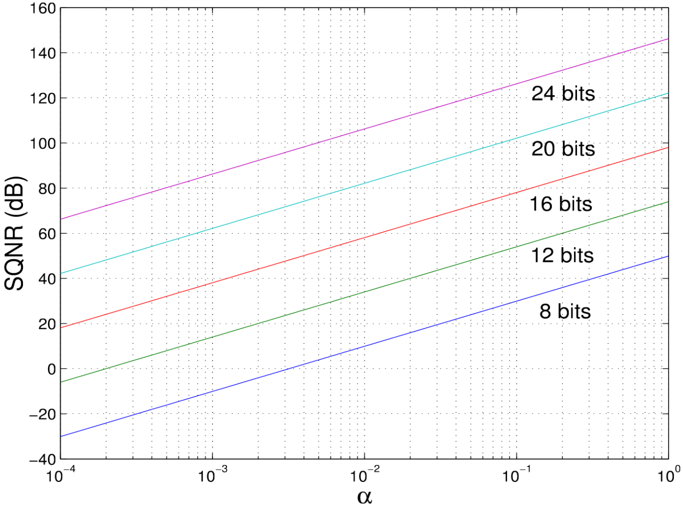
Assume $v(t)$ is sinusoidal:

$$P_v = \frac{A^2}{2} = \frac{1}{2}\left(\frac{\alpha R}{2}\right)^2 = \frac{\alpha^2 R^2}{8}$$



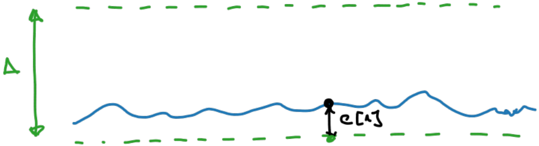
$$\mathrm{SQNR} = 6.02B + 20\log_{10}\alpha + 1.76$$

SQNR for sinusoidal signals

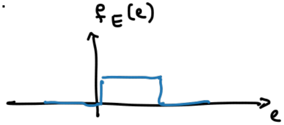


Is the uniform noise levels always a good assumption?

Consider this case, zooming in between two of the quantisation intervals:

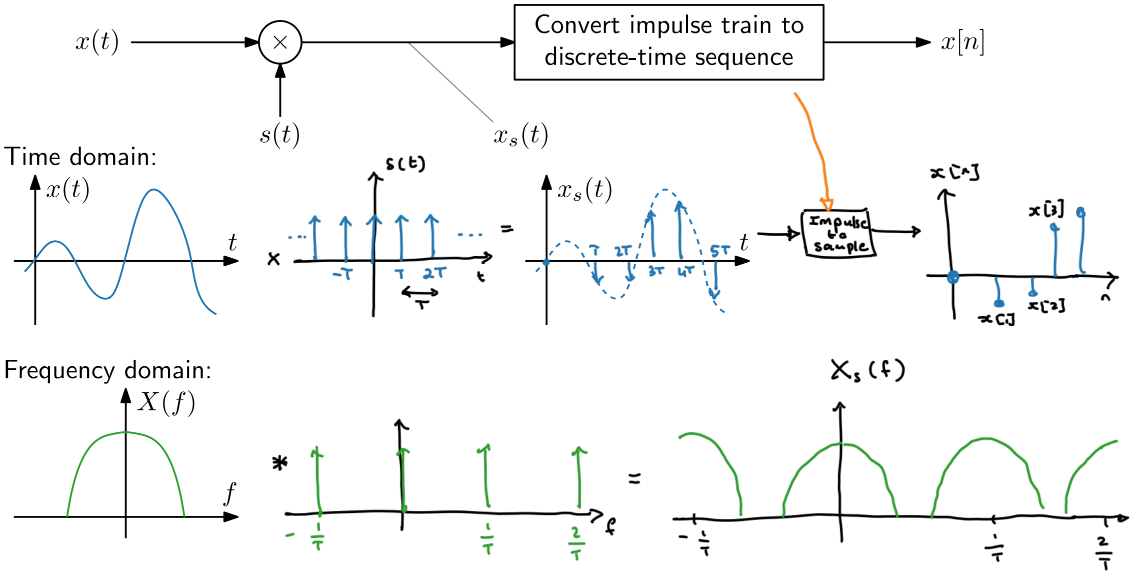


Now the noise distribution is closer to:

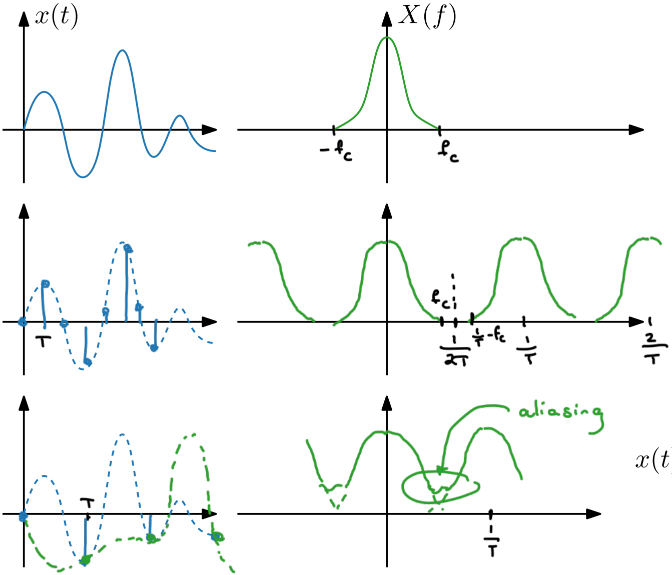


Moral of the story: Is your model correct?

Discretising in time



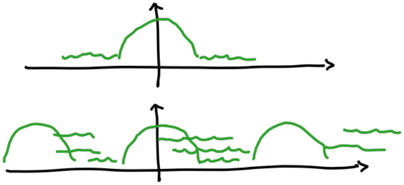
Aliasing

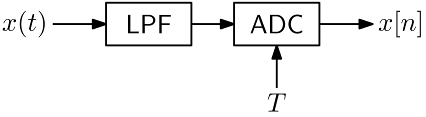


Nyquist: $f_c \leq \frac{1}{2T} = \frac{f_s}{2}$

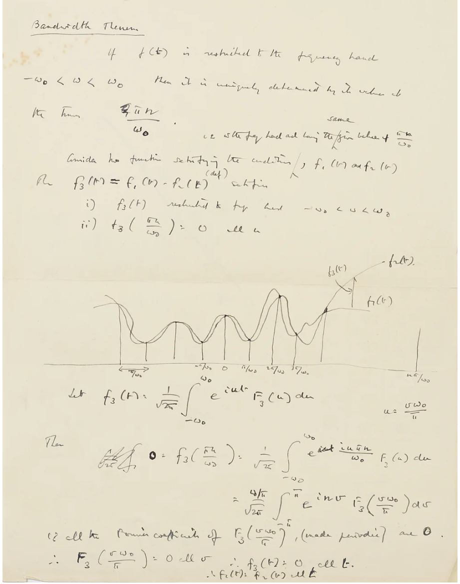
$f_s \geqslant 2 f_c$

Good to have aliasing filter, even if band-limited, because of noise:





Alan Turing notes from the Bayley papers



Bandwidth Theorem

If $f(t)$ is restricted to the frequency band $-\omega_0 < \omega < \omega_0$ then it is uniquely determined by its values at the times $\frac{\pi n}{\omega_0}$.

i.e. with freq band and having the same values at $\frac{\pi n}{\omega_0}$

Consider two functions satisfying the conditions (def), $f_1(t)$ and $f_2(t)$

Then $f_3(t) = f_1(t) - f_2(t)$ satisfies

i) $f_3(t)$ restricted to freq band $-\omega_0 < \omega < \omega_0$

ii) $f_3\left(\frac{\pi n}{\omega_0}\right) = 0$ all n

Let $f_3(t) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_0}^{\omega_0} e^{i\omega t} F_3(\omega)\, d\omega$ $\qquad \omega = \frac{v\omega_0}{\pi}$

Then $0 = f_3\left(\frac{\pi n}{\omega_0}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_0}^{\omega_0} e^{\frac{i\omega\pi n}{\omega_0}} F_3(\omega)\, d\omega$

$= \frac{\omega_0/\pi}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{inv} F_3\left(\frac{v\omega_0}{\pi}\right) dv$

i.e. all the Fourier coefficients of $F_3\left(\frac{v\omega_0}{\pi}\right)$ (made periodic) are 0.

$\therefore F_3\left(\frac{v\omega_0}{\pi}\right) = 0$ all v $\quad \therefore f_3(t) = 0$ all t.

$\therefore f_1(t) = f_2(t)$ all t

Alan Turing notes from the Bayley papers

Among the piles of loose sheets in the Baley papers, one page is headed “Bandwidth Theorem”. The notes contain a version of what is now called the Nyquist–Shannon sampling theorem. Turing most likely wrote this out as a teaching aid for his young assistant, Donald Bayley, who was an electrical engineer drafted to work with him on developing a secure speech-encryption device during World War II. Bayley kept many of these wartime papers, which show Turing explaining mathematical principles and practical circuit techniques to him.

Claude Shannon had drafted a paper reviewing earlier work on the theorem and presenting his own version of it. Although written in 1940, the paper was not published until 1949. Turing spent time at Bell Labs in 1943 working on SIGSALY before returning to Britain. It is therefore quite plausible that he and Shannon discussed issues of sampling rates during this period.

Based on the IEEE Spectrum article by Copeland (2025).