Linear time-invariant (LTI) systems

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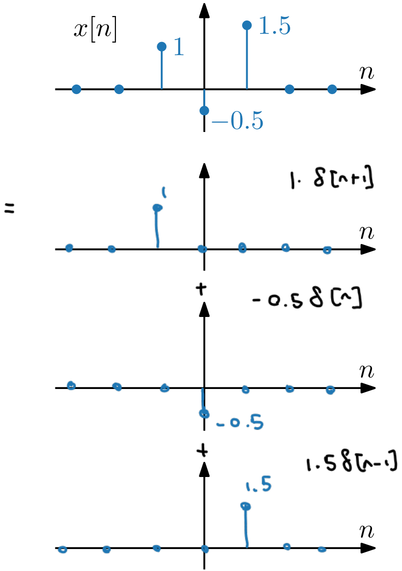


Any discrete signal: $x[n] = \sum_{i=-\infty}^{\infty} x[i]\cdot\delta[n-i]$

Discrete system: $y[n] = \mathcal{T}\{x[n]\}$

$$= \mathcal{T}\left\{\sum_{i=-\infty}^{\infty} x[i]\cdot\delta[n-i]\right\}$$

Assume linear: $y[n] = \sum_{i=-\infty}^{\infty} x[i]\cdot\mathcal{T}\{\delta[n-i]\}$

Assume time-invariant: $h[n] = \mathcal{T}\{\delta[n]\}$ Called "impulse response"

$$\mathcal{T}\{\delta[n-i]\} = h[n-i]$$

Linear time-invariant (LTI) system:

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]\cdot h[n-i] = x[n] * h[n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

LTI example

$$h[n] = \{\underset{\uparrow}{2} \quad 3 \quad 1 \quad 2\}$$

$$x[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3 \quad 1\}$$

What is $y[n]$?

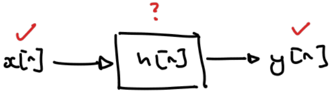
$$h[i] = \{\underset{\uparrow}{2} \quad 3 \quad 1 \quad 2\}$$

$$n=0: \quad x[-i] = \{1 \quad 3 \quad 2 \quad \underset{\uparrow}{1}\}$$

$$n=1: \quad x[1-i] = \{1 \quad 3 \quad \underset{\uparrow}{2} \quad 1\}$$

$$y[n] = x[n] * h[n] = \{\underset{\uparrow}{2} \quad 7 \quad 13 \quad 15 \quad 10 \quad 7 \quad 2\}$$

What if I gave you $x[n]$ and $y[n]$, could you then figure out $h[n]$?
This is part of "system identification"



Causality in LTI systems

$$y[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

$$y[\overset{10}{n_0}] = \sum_{i=-\infty}^{\infty} h[i]\cdot x[\overset{10}{n_0}-i]$$

$x[10]$ ✓

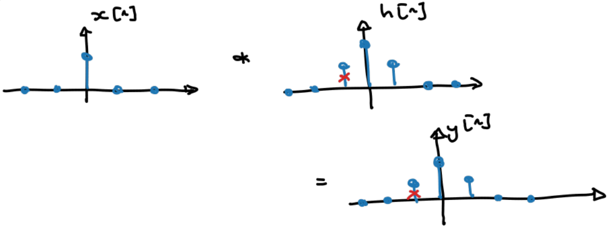
$x[11]$ ✗

Causal system: $y[n_0]$ only depends on $x[n_0-i]$ where $i \geq 0$

$$y[n_0] = \overbrace{\dots + h[-2]\cdot x[n_0+2] + h[-1]\cdot x[n_0+1]}^{\times} + \underbrace{h[0]\,x[n_0] + h[1]\,x[n_0-1] + \dots}_{\checkmark}$$

$$= \underbrace{\sum_{i=-\infty}^{-1} h[i]\,x[n_0-i]}_{=0} + \sum_{i=0}^{\infty} h[i]\,x[n_0-i]$$

$\therefore\ h[n] = 0$ for $n < 0$



Causal LTI system:

$$\boxed{h[i] = 0 \text{ for all } i < 0}$$

If we also have that $x[n] = 0$ for $n < 0$, then:

$$\begin{aligned} y[n] &= \sum_{i=0}^{n} h[i]x[n-i] \\ &= \sum_{i=0}^{n} x[i]h[n-i] \end{aligned}$$

Stability of LTI systems

BIBO

$$y[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

$$|y[n]| = \left| \sum_{i=-\infty}^{\infty} h[i] \cdot x[n-i] \right| = \left| \ldots + h[-2] \cdot x[n+2] + h[-1]\,x[n+1] + h[0]\,x[n] + h[1]\,x[n-1] + \ldots \right|$$

$$\leq \ldots + |h[-2]| \cdot |x[n+2]| + |h[-1]| \cdot |x[n+1]| + |h[0]| \cdot |x[n]| + \ldots$$

$$= \sum_{i=-\infty}^{\infty} |h[i]|\,|x[n-i]|$$

$|x[n]| \leq M_x$ for all n

$$\leq \sum_{i=-\infty}^{\infty} |h[i]|\,M_x = M_x \cdot \sum_{i=-\infty}^{\infty} |h[i]|$$

$\therefore$ BIBO stable if $\underbrace{\text{and only}}_{}$ if $\sum_{i=-\infty}^{\infty} |h[i]| < \infty$

We didn't prove this

See note on "Necessity, sufficiency and stability"

Above we proved that having a impulse response that is absolutely summable is *sufficient* for an LTI system to be BIBO stable. But is it also *necessary*? Can we have an impulse reponse (maybe a very special one) that sums to infinity but gives bounded output for any input? Our proof says nothing about this so far. But absolute summability turns out to be both sufficient and necessary.

Proof of necessity. Strategy: we find an input for which the condition is strictly necessary.

$$x[n-i] = \begin{cases} +1 & \text{if } h[i] > 0 \\ 0 & \text{if } h[i] = 0 \\ -1 & \text{if } h[i] < 0 \end{cases}$$

which means that for any fixed value of n you have:

$$\begin{aligned} |y[n]| &= \left| \sum_{i=-\infty}^{\infty} h[i]x[n-i] \right| \\ &= \sum_{i=-\infty}^{\infty} |h[i]| \end{aligned}$$

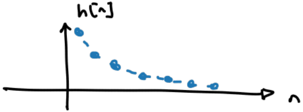
This means that, for $y[n]$ to be bounded, you *need* $h[n]$ to be absolutely summable.

An LTI system is stable if and only if its impulse response is summable:

$$\boxed{\sum_{i=-\infty}^{\infty} |h[i]| < \infty}$$

From this result it can be shown that:

- $|h[n]| \to 0$ as $n \to \infty$
- $|y[n]| \to 0$ as $n \to \infty$ for finite-duration $x[n]$



Subclasses of LTI systems

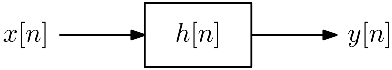
Discrete systems in general

↓

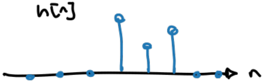
LTI systems

↓

Types of LTI systems

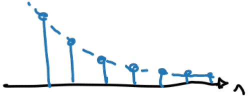

$x[n]$ → $h[n]$ → $y[n]$

$y[n] = x[n] * h[n]$



- Finite impulse response (FIR)
  (moving average)
- Infinite impulse response (IIR)

E.g. decaying exponential

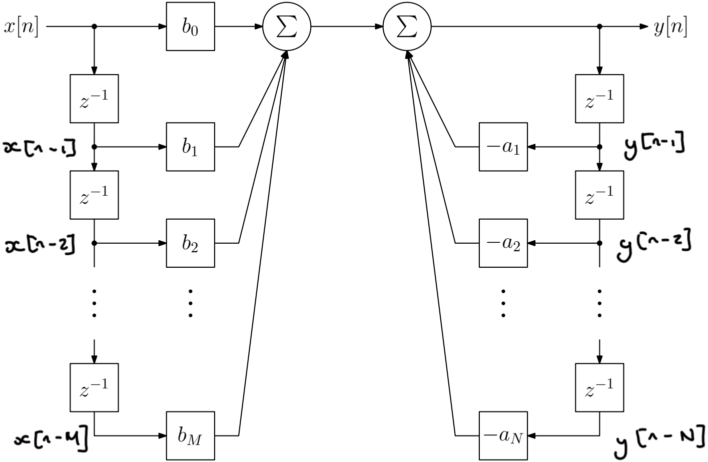


- Linear constant-coefficient difference equation (LCCDE): Output is linear combination of finite number of weighted past outputs and past and present inputs

Linear constant-coefficient difference equation (LCCDE)

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

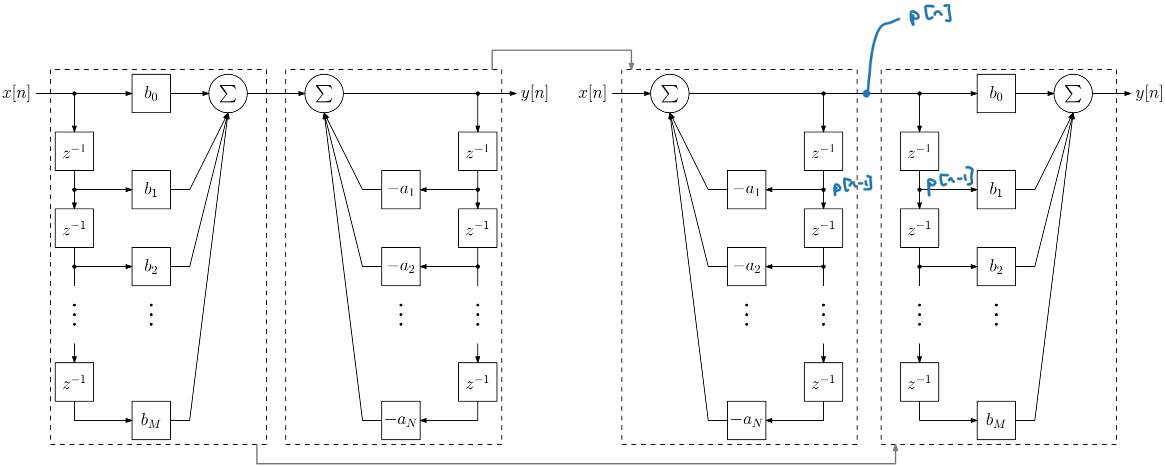
Moving average



Autoregressive

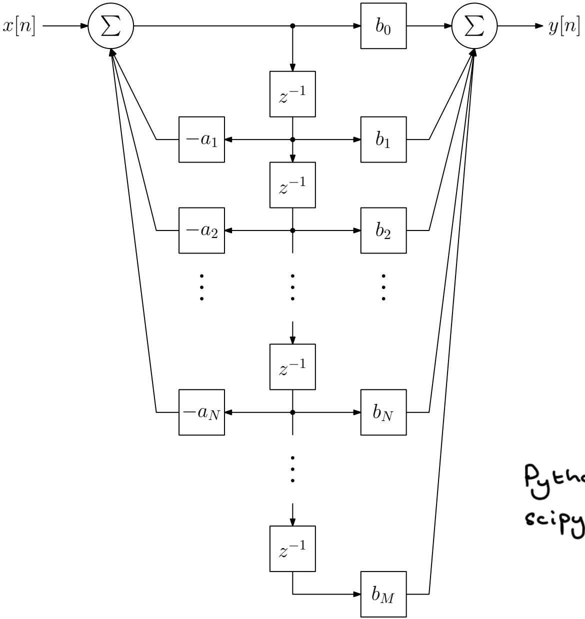
DIRECT FORM I

Efficient LCCDE implementation



$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n] \equiv x[n] \rightarrow \boxed{h_2[n]} \rightarrow \boxed{h_1[n]} \rightarrow y[n]$$

$$= (x[n] * h_1[n]) * h_2[n]$$



Python:
scipy.signal.lfilter

DIRECT FORM II

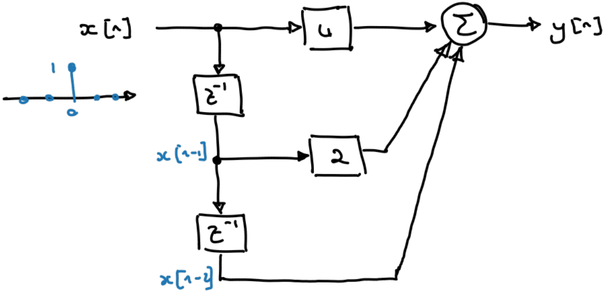
LCCDE example

$$y[n] = 4x[n] + 2x[n-1] + x[n-2]$$

(a) Draw the direct-form I for this filter

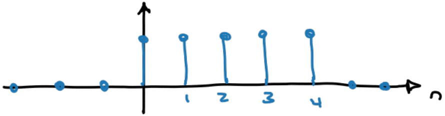
(b) What is the impulse response of this filter?

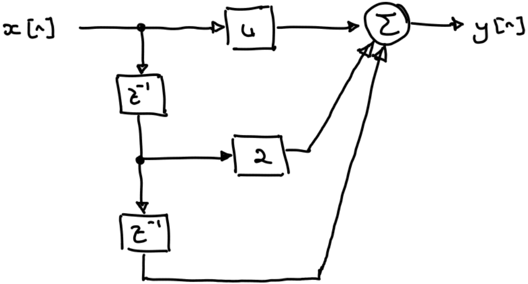
$$h[n] = \{ \underset{\uparrow}{4} \quad 2 \quad 1 \}$$



$$y[n] = 4x[n] + 2x[n-1] + x[n-2]$$

(c) What is the filter's output for $x[n] = u[n] - u[n-5]$?





$$y[n] = \{\underset{\uparrow}{4} \quad 6 \quad 7 \quad 7 \quad 7 \quad 3 \quad 1\}$$

Can do from block diagram

or from $y[n] = x[n] * h[n]$

Obtained on previous slide