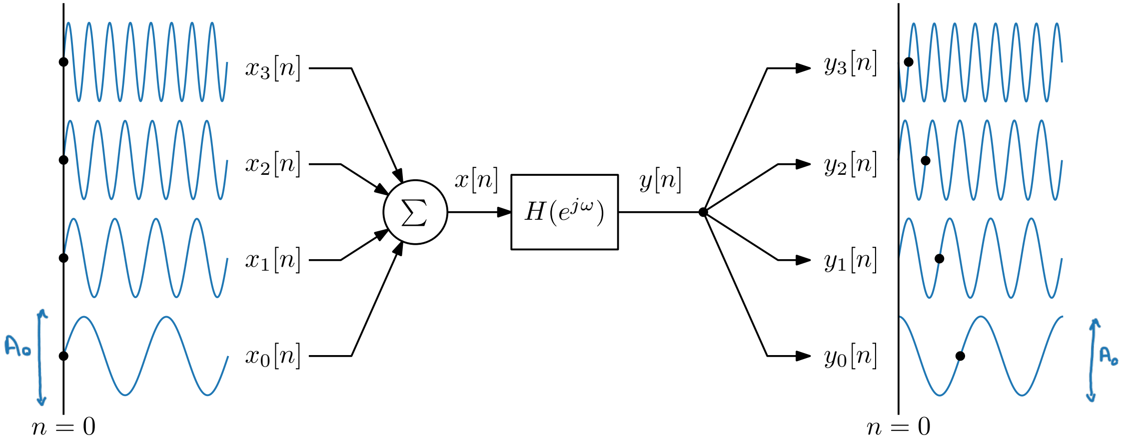
Filter phase characteristics

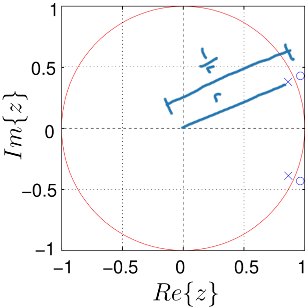
All-pass filters and linear phase

Herman Kamper

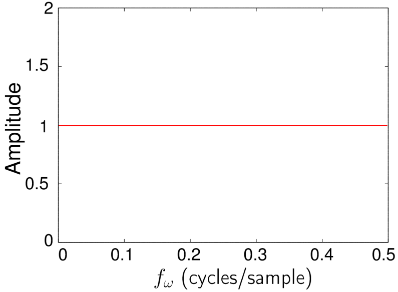
Filter phase characteristics

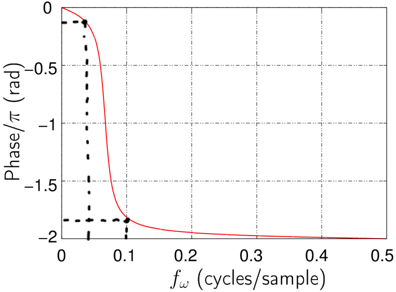


$$H(z) = \frac{0.9 - \sqrt{3}z^{-1} + z^{-2}}{1 - \sqrt{3}z^{-1} + 0.9z^{-2}}$$



Feed with input:

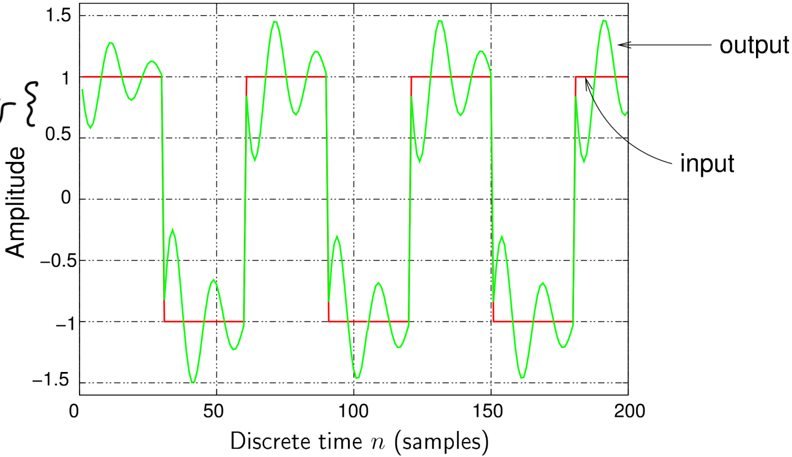




All-pass filter input and output:

To your ears, these will sound the same

Problem if you e.g. use thresholding



Linear phase

$$x[n] = \sum_{i=0}^{K} x_i[n] \longrightarrow \boxed{h[n]} \longrightarrow$$

What we want:

$$y_i[n] = A_i\, x_i[n-n_0] \quad \ldots \text{①}$$

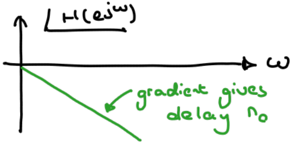
and n_0 should be the same for all i

$$x_i[n] = e^{j\omega_i n} \quad \ldots \text{②}$$

then
$$y_i[n] = |H(e^{j\omega_i})|\, e^{j(\omega_i n + \angle H(e^{j\omega_i}))}$$
$$= A_i\, e^{j(\omega_i n + \phi_i)}$$

For ① to be true:
$$x_i[n-n_0] = e^{j(\omega_i n + \phi_i)}$$
$$e^{j\omega_i(n-n_0)} = e^{j(\omega_i n + \phi_i)} \quad \text{②}$$
$$j\omega_i n - j\omega_i n_0 = j\omega_i n + j\phi_i$$
$$\therefore \phi_i = -\omega_i n_0$$
$$\angle H(e^{j\omega}) = -\omega n_0$$

We want linear phase :



All-pass filters

$$\left|H(e^{j\omega})\right| = 1 \text{ for all } \omega$$

E.g. $A(z) = z^2 + z - 2 = (z+2)(z-1)$ $\quad a = z^{-1}$

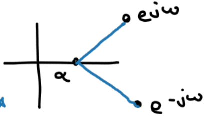
$A'(z) = z^{-2} + z^{-1} - 1 = a^2 + a - 1$

$= (a+2)(a-1)$

$= (z^{-1}+2)(z^{-1}-1)$

$$A(z) = z^N + a_{N-1}z^{N-1} + a_{N-2}z^{N-2} + \ldots + a_1 z + a_0$$
$$A'(z) = z^{-N} + a_{N-1}z^{-(N-1)} + a_{N-2}z^{-(N-2)} + \ldots + a_1 z^{-1} + a_0$$

Easiest to see by vector length



For every factor $(z-\alpha)$ in $A(z)$

we have $(z^{-1}-\alpha)$ in $A'(z)$

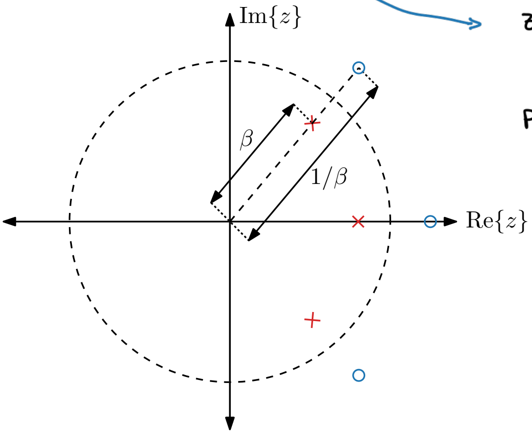
For real α: $\left|z-\alpha\right|_{z=e^{j\omega}} = \left|e^{j\omega}-\alpha\right| = \left|z^{-1}-\alpha\right|_{z=e^{j\omega}} = \left|e^{-j\omega}-\alpha\right|$

For complex factors: $\Big|\left|z^{-1}-\alpha\right|\left|z^{-1}-\alpha^*\right|\Big|_{z=e^{j\omega}} = \Big|\left|z-\alpha\right|\left|z-\alpha\right|\Big|_{z=e^{j\omega}}$

$$\therefore \left|\frac{A'(z)}{A(z)}\right|_{z=e^{j\omega}} = 1 \quad \text{for all } \omega$$

$$H(z) = \frac{A'(z)}{A(z)/z^N} = \frac{z^{-N} + a_{N-1}z^{-(N-1)} + a_{N-2}z^{-(N-2)} + \ldots + a_1 z^{-1} + a_0}{1 + a_{N-1}z^{-1} + a_{N-2}z^{-2} + \ldots + a_1 z^{-(N-1)} + a_0 z^{-N}}$$

$$= \prod_{k=0}^{N_r} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=0}^{N_c} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})}$$



Zero : $z^{-1} - \beta_k = 0$

Zero $= \frac{1}{\beta_k}$

Pole : $1 - \beta_k z^{-1} = 0$

$1 - \frac{\beta_k}{z} = 0$

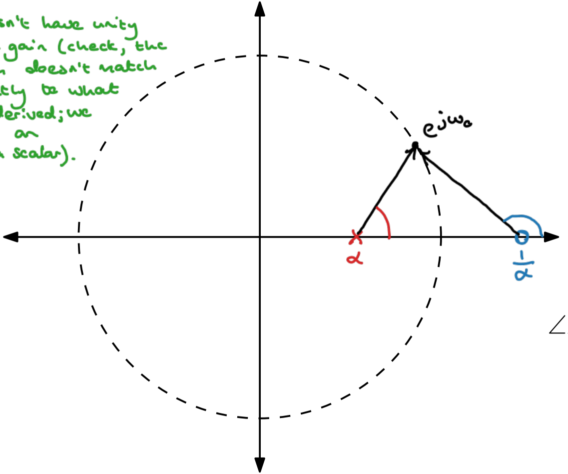
Pole $= \beta_k$

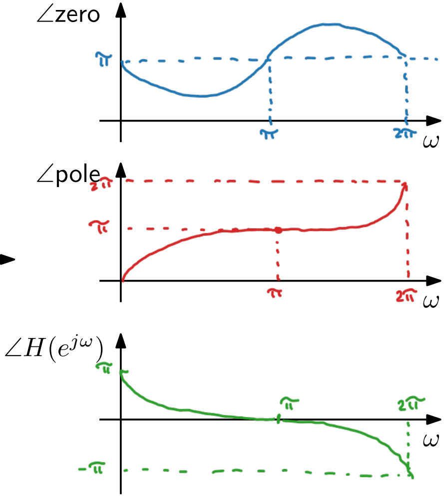
All-pass filter example

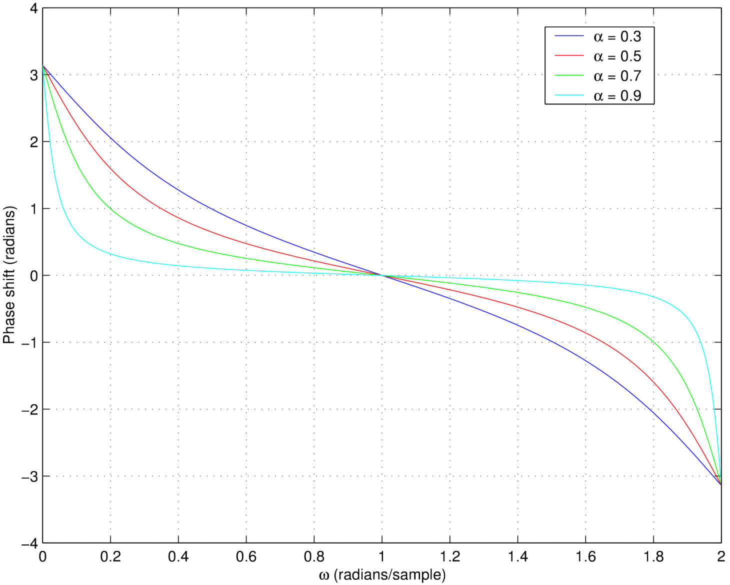
$$\angle H(e^{j\omega}) = \angle \text{zeros} - \angle \text{poles}$$

$$H(z) = \frac{1 - \frac{1}{\alpha} z^{-1}}{1 - \alpha z^{-1}}$$

Doesn't have unity DC-gain (check, the form doesn't match exactly to what we derived; we need an extra scalar).







α = 0.3
α = 0.5
α = 0.7
α = 0.9
Phase shift (radians)
ω (radians/sample)
4
3
2
1
0
−1
−2
−3
−4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Practical applications of all-pass filters

- Although our ears ignore phase to a large degree, we have two ears
- E.g. in a car with multiple speakers, phase could affect our preception of the audio (Bellini et al. 2001)
- Watch from 6:40 to around 11:00 of All pass filters and when to use them?
- Used in chains of filters for audio effects, e.g. a phaser
- Improved feedback loop stability in control systems (Yao et al. 2018)
- Used to fix problems with phase