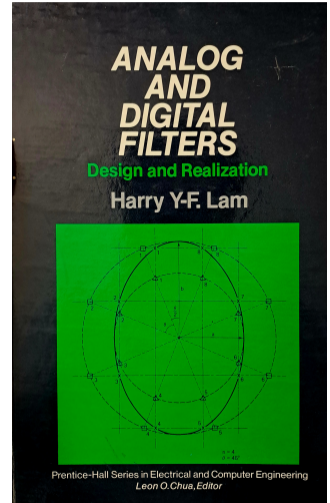


Infinite impulse response (IIR) filter design

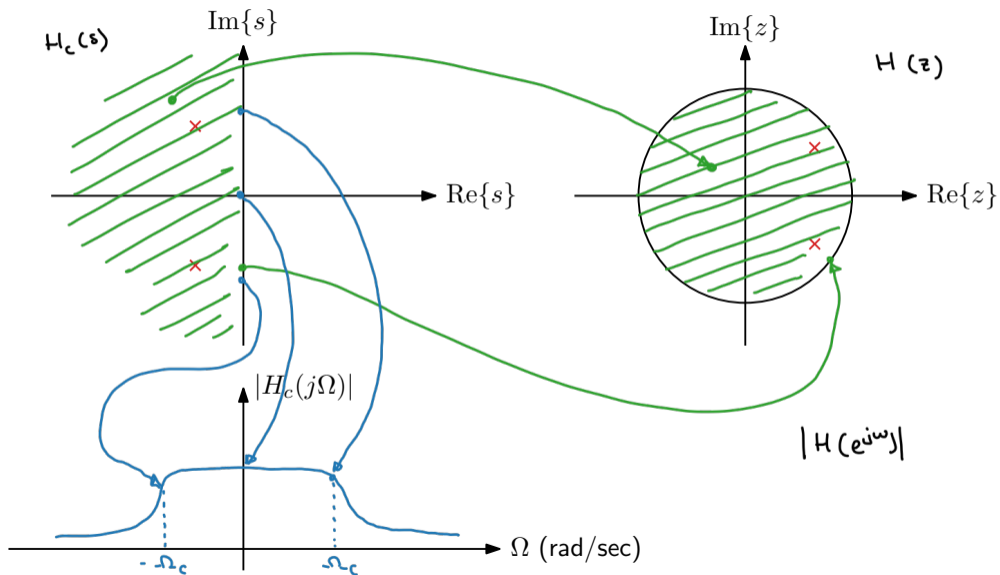
Herman Kamper

Discrete filter design methods

- Place poles and zeros
- Hack the ideal impulse response to make it realisable (FIR)
- Convert continuous filters to discrete filters (IIR)



Mapping the s-plane to the z-plane



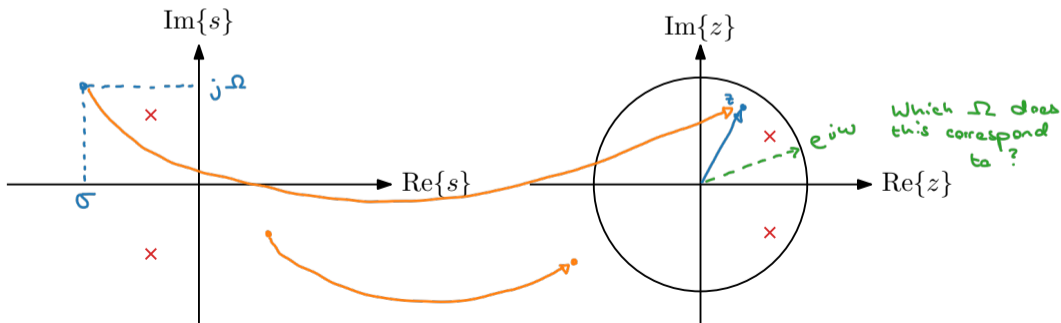
Bilinear transform

$$s = \frac{1}{K} \frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{and} \quad z = \frac{1 + Ks}{1 - Ks} \quad \text{with } K > 0$$

Let $s = \sigma + j\Omega$:

$$z = \frac{1 + K(\sigma + j\Omega)}{1 - K(\sigma + j\Omega)} = \frac{1 + K\sigma + jK\Omega}{1 - K\sigma - jK\Omega} \Rightarrow |z|^2 = \frac{(1 + K\sigma)^2 + K^2\Omega^2}{(1 - K\sigma)^2 + K^2\Omega^2}$$

- What is $|z|$ if $\sigma = 0$? $|z| = 1$
- What is $|z|$ if $\sigma < 0$? $|z| < 1$



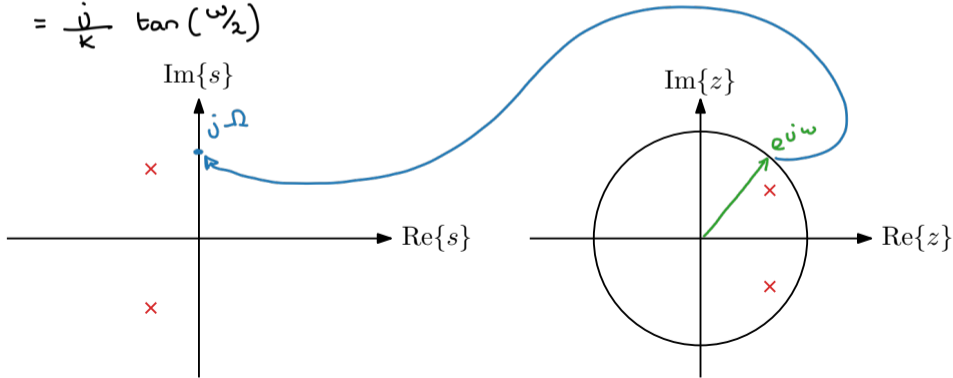
Let $z = e^{j\omega}$:

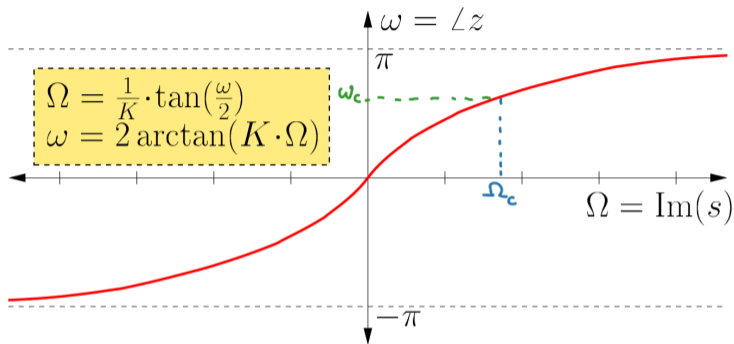
$$\begin{aligned}
 s &= \frac{1}{K} \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \\
 &= \frac{1}{K} \frac{\cancel{e^{-j\omega/2}} (e^{j\omega/2} - e^{-j\omega/2})}{\cancel{e^{-j\omega/2}} (e^{j\omega/2} + e^{-j\omega/2})} \cdot \frac{z}{2} \cdot \frac{2j}{2j} \\
 &= \frac{1}{K} \frac{\sin(\omega/2)}{\cos(\omega/2)} \frac{2j}{2} \\
 &= \frac{j}{K} \tan(\omega/2)
 \end{aligned}$$

$$s = \frac{1}{K} \frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{and} \quad z = \frac{1 + Ks}{1 - Ks}$$

$$\therefore \Omega = \frac{1}{K} \tan\left(\frac{\omega}{2}\right)$$

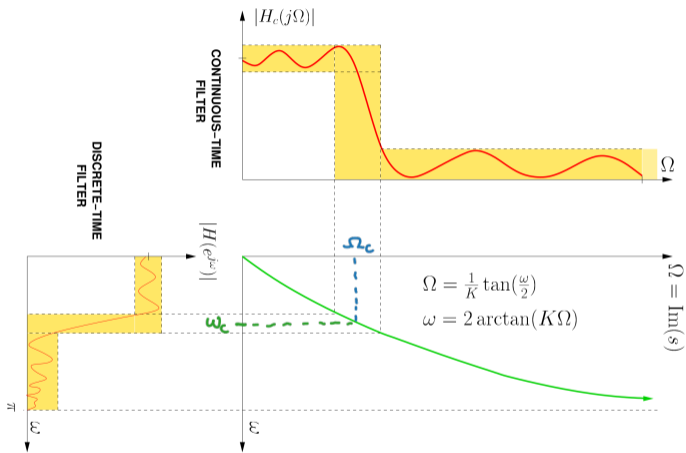
$$\therefore \omega = 2 \arctan(\Omega K)$$





Maps: $\Omega \in (-\infty, \infty)$ to $\omega \in (-\pi, \pi)$

Continuous to discrete filter design procedure



1. Specification in discrete time
2. Pre-warp specification frequencies $\Omega = \frac{1}{K} \tan\left(\frac{\omega}{2}\right)$
3. Design continuous-time filter to pre-warped specification $H_c(s)$
4. Substitute bilinear transform $s = \frac{1}{K} \frac{1-z^{-1}}{1+z^{-1}}$

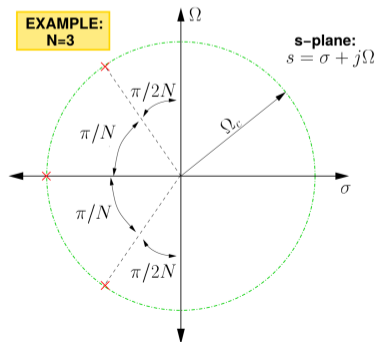
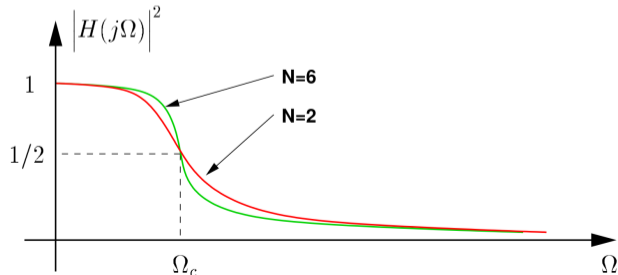
Continuous-time Butterworth filter

Butterworth LPF of order N has magnitude response:

$$|H_c(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

with N poles at

$$s_k = \Omega_c e^{j\pi \frac{N+1+2k}{2N}} \text{ for } k = 0, 1, \dots, N-1$$



Butterworth discrete filter example

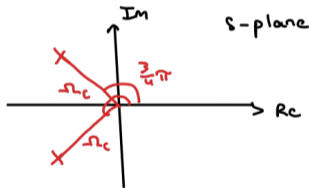
Use a second-order analog Butterworth LPF as a prototype to design a discrete LPF with $\omega_c = 0.2\pi$ rad/sample.

1. Specification in discrete time $\omega_c = 0.2\pi$ rad/sample
2. Pre-warp specification frequencies $\Omega = \frac{1}{K} \tan\left(\frac{\omega}{2}\right)$ $\Omega_c = \frac{1}{K} \tan\left(\frac{0.2\pi}{2}\right)$
 $= 0.325$ rad/sec
3. Design continuous-time filter to pre-warped specification

$$N=2: s_k = \Omega_c e^{j\pi \frac{N+1+2k}{2N}} \quad \text{for } k=0, 1$$
$$s_0 = \Omega_c e^{j\pi \frac{2+1}{2 \cdot 2}} = \Omega_c e^{j\frac{3}{4}\pi}$$
$$s_1 = \Omega_c e^{j\frac{5}{4}\pi}$$

Ω_c^2
(unity gain)

$$H_c(s) = \frac{\Omega_c^2}{(s - \Omega_c e^{j\frac{3}{4}\pi})(s - \Omega_c e^{j\frac{5}{4}\pi})}$$
$$= \frac{\Omega_c^2}{s^2 + \sqrt{2} \Omega_c s + \Omega_c^2}$$



4. Substitute bilinear transform $s = \frac{1}{K} \frac{1-z^{-1}}{1+z^{-1}}$

$$H_c(s) = \frac{\Omega_c^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$

$$H(z) = H_c(s) \Big|_{s = \frac{1}{K} \frac{1-z^{-1}}{1+z^{-1}}} \quad K=1$$

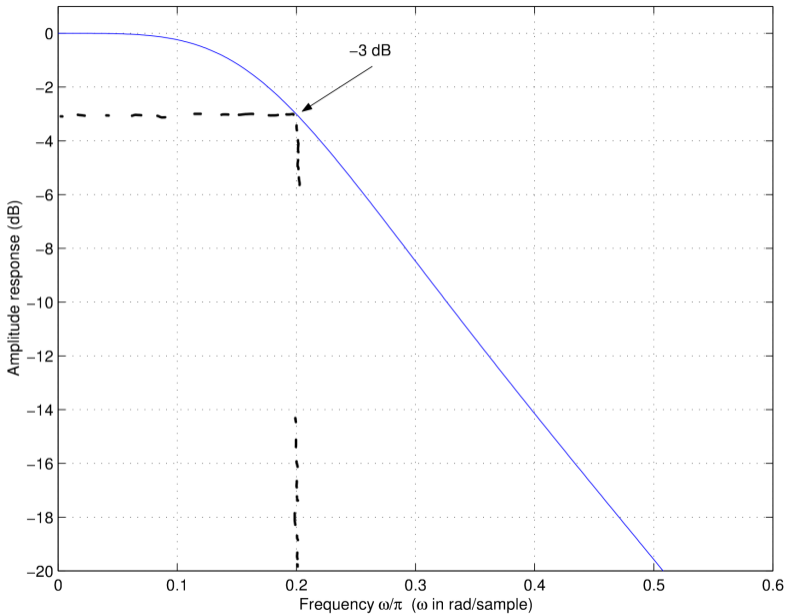
$$= \frac{\Omega_c^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \sqrt{2}\Omega_c \left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \Omega_c^2}$$

⋮

$$= \frac{0.06773 + 0.135z^{-1} + 0.0677z^{-2}}{1 - 1.143z^{-1} + 0.413z^{-2}}$$

$\Omega_c = 0.325$

Frequency response of bilinear-transformed 2nd-order Butterworth filter with $\omega_c = 0.2\pi$ rad/sample



Pole-zero plot for bilinear-transformed 2nd-order Butterworth filter with $\omega_c = 0.2\pi$ rad/sample

