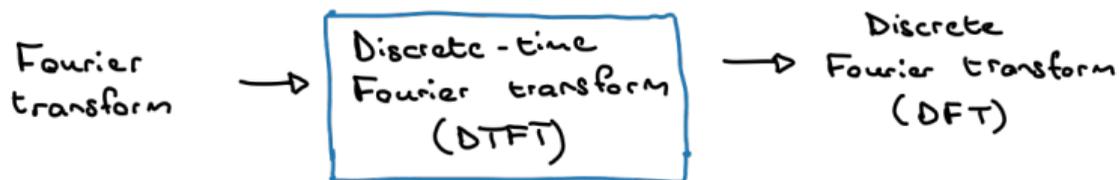


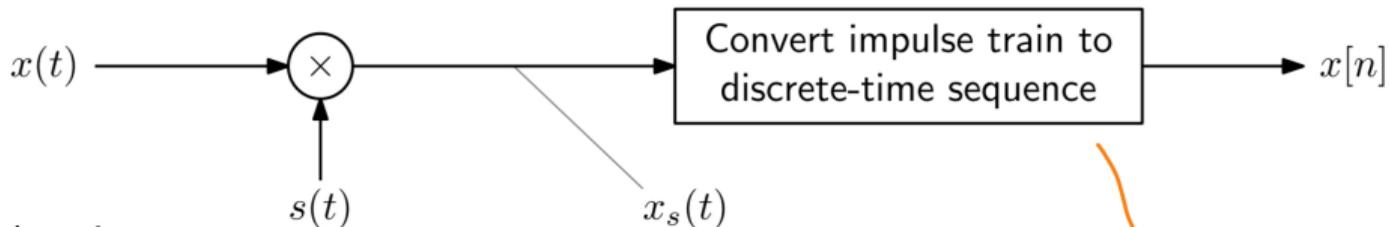
Discrete-time Fourier transform (DTFT)

And a case study on sampled sinusoids

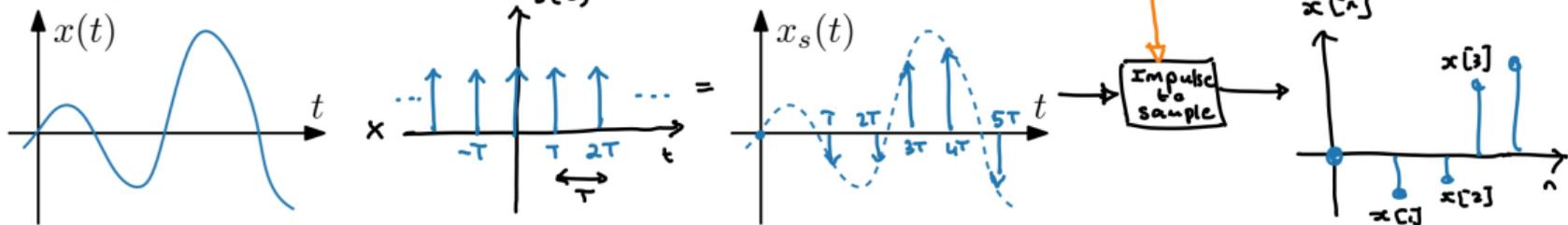
Herman Kamper



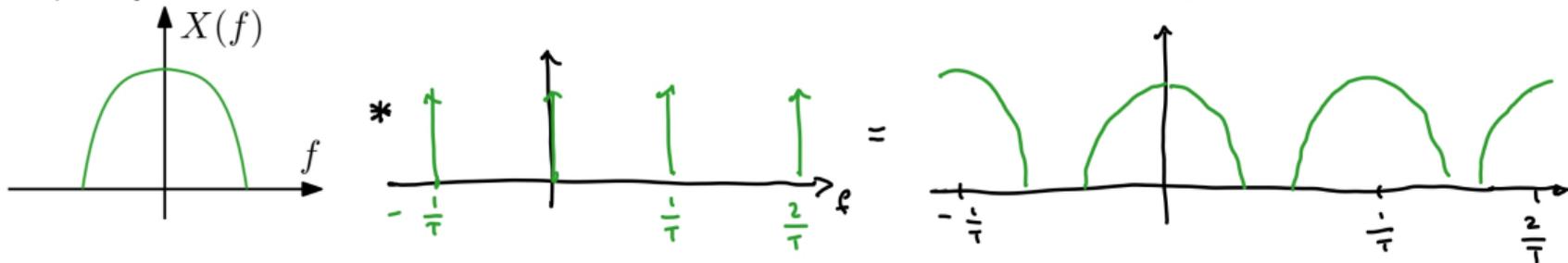
Mathematical model of sampling



Time domain:



Frequency domain:



Discrete-time Fourier transform (DTFT)

$$x_s(t) = x(t) \cdot s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t-nT)$$

$$\begin{aligned} X_s(f) &= \mathcal{F}\{x_s(t)\} = \int_{-\infty}^{\infty} x_s(t) \cdot e^{-j2\pi ft} \cdot dt \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t-nT) \right] \cdot e^{-j2\pi ft} \cdot dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(t-nT) \cdot e^{-j2\pi ft} \cdot dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) \cdot e^{-j2\pi f n T} = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f n T} \end{aligned}$$

Define:

$$f_w \equiv fT = \frac{f}{f_s}$$

f : [cycles/sec]
 f_s : [samples/sec]
 f_w : [cycles/sample]

f_w is called
discrete-time frequency
or
normalised frequency

DTFT:

$$X(f_w) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f_w n} \quad [\text{rad/sample}]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} \quad \text{with } \omega = 2\pi f_w$$

Periodicity:

$$X(f_w) = X(f_w + k)$$

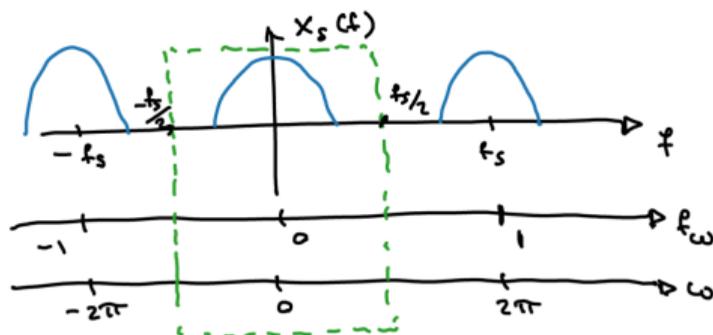
$$X(\omega) = X(\omega + 2\pi k)$$

integer

Inverse discrete-time Fourier transform (IDTFT)

Define $\hat{X}_s(f)$ to correspond to a single period of $X_s(f)$:

$$\hat{X}_s(f) = \begin{cases} X_s(f) & \text{for } -\frac{f_s}{2} \leq f \leq \frac{f_s}{2} \\ 0 & \text{otherwise} \end{cases}$$



The original $X_s(f)$ can be recovered by convolving $\hat{X}_s(f)$ with an impulse train:

$$X_s(f) = \hat{X}_s(f) * \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

Therefore we can write the inverse Fourier transform of $X_s(f)$ as

$$\mathcal{F}^{-1}\{X_s(f)\} = \mathcal{F}^{-1}\{\hat{X}_s(f)\} \cdot \mathcal{F}^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(f - kf_s)\right\}$$

The first term in the last equation is

$$\mathcal{F}^{-1}\{\hat{X}_s(f)\} = \int_{-\infty}^{\infty} \hat{X}_s(f) e^{j2\pi ft} df = \int_{-f_s/2}^{f_s/2} X_s(f) e^{j2\pi ft} df$$

and the second term is

$$\mathcal{F}^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(f - kf_s)\right\} = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_s}\right)$$

Combining these, we obtain:

$$x_s(t) = \mathcal{F}^{-1}\{X_s(f)\} = \left[\int_{-f_s/2}^{f_s/2} X_s(f) e^{j2\pi ft} df \right] \cdot \left[\frac{1}{f_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_s}\right) \right]$$

This equation describes a continuous function sampled by multiplication with an impulse train with an impulse every $1/f_s = T$ seconds. The sequence $x[n]$ is the strength of these impulses:

$$x[n] = x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f) e^{j2\pi fnT} df$$

We therefore have:

$$x[n] = x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f) e^{j2\pi f n T} df$$

Converting the integration over variable f so that we instead integrate over $f_\omega = fT$, we obtain the inverse DTFT:

$$\begin{aligned} x[n] &= \int_{-1/2}^{1/2} X(f_\omega) e^{j2\pi f_\omega n} df_\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \end{aligned}$$

$$f_\omega = fT = \frac{f}{f_s}$$

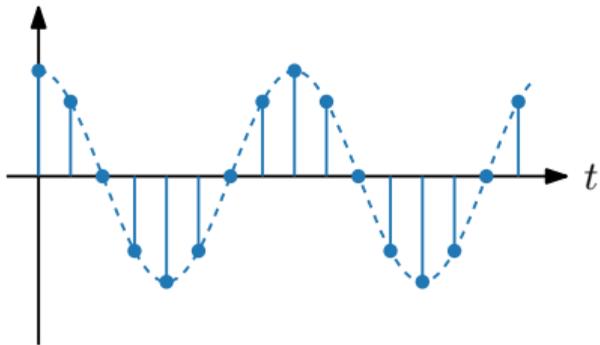
$$\frac{df_\omega}{df} = \frac{1}{f_s}$$

$$\therefore df_\omega = \frac{1}{f_s} df$$

f	f_ω
$-1/2$	$-1/2$
$1/2$	$1/2$

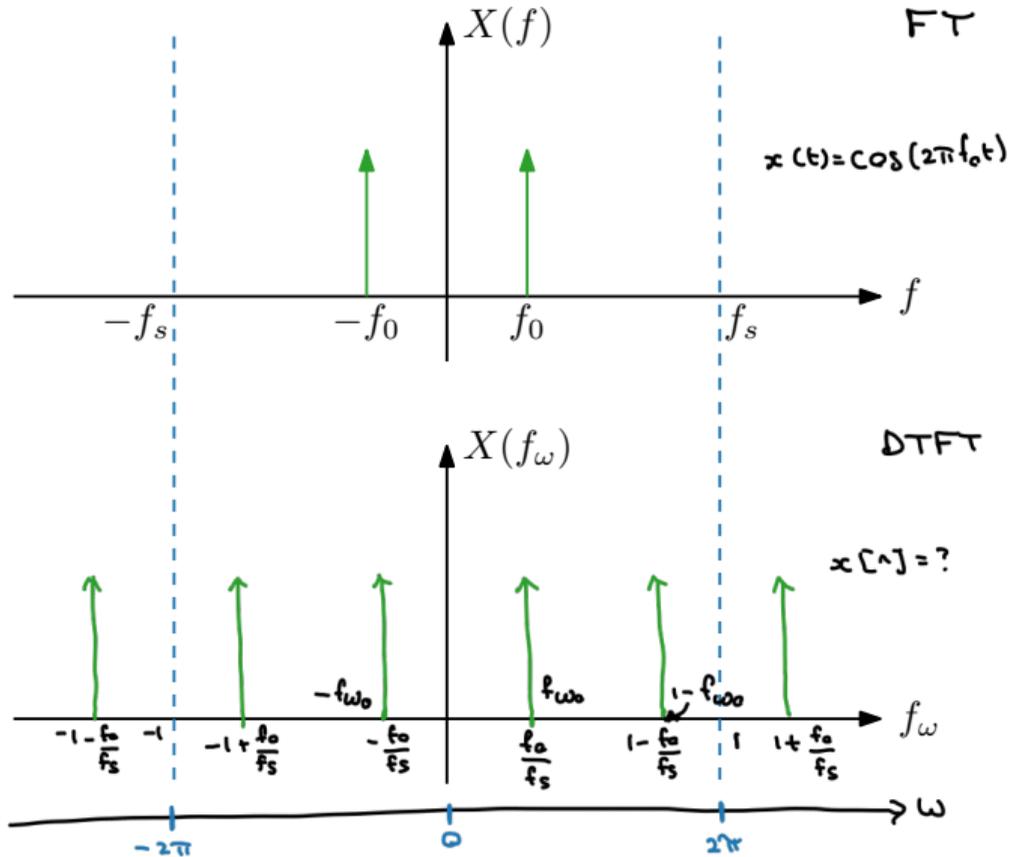
The second version is just expressed in terms of discrete angular frequency $\omega = 2\pi f_\omega$.

Case study: DTFT of a sinusoid



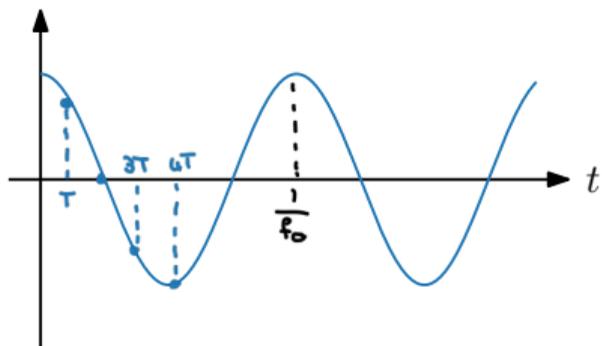
$$f_{\omega} = \frac{f}{f_s}$$

$$\omega = 2\pi f_{\omega}$$



Continuous vs discrete-time frequency

$$x(t) = \cos(2\pi f_0 t)$$



$$x(t) = \cos(2\pi f_0 t)$$

$$x(nT) = \cos(2\pi f_0 nT)$$

$$= \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$$\therefore x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$$x[n] = \cos(2\pi f_{\omega_0} n)$$

$$f_{\omega_0} = \frac{f_0}{f_s}$$

Continuous

f_0 : continuous cycles/sec

Ω_0 : rad/sec

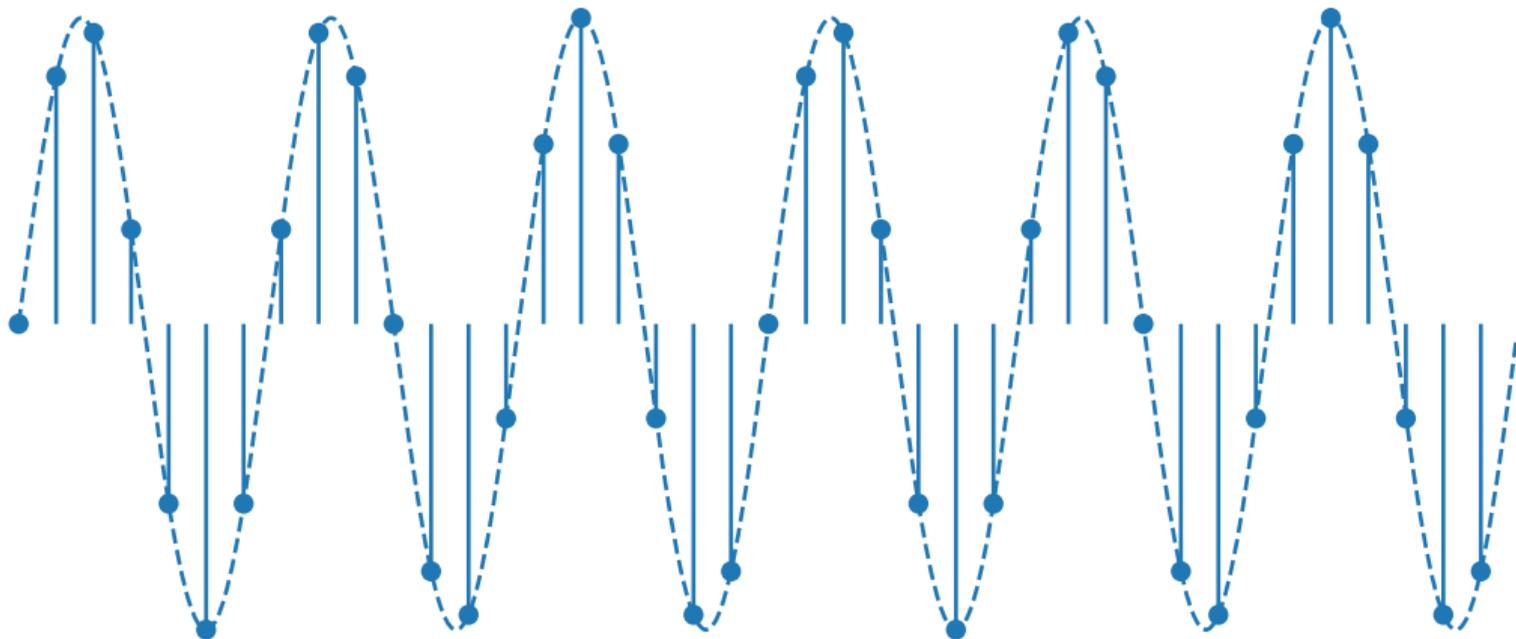
Discrete

f_{ω_0} : continuous cycles/sample

ω_0 : rad/sample

300 Hz signal sampled at 2000 Hz:

$$f_{\omega_0} = \frac{f_0}{f_s} = \frac{300}{2000} = \frac{3}{20} \quad \therefore x[n] = \sin\left(2\pi \cdot \frac{3}{20} n\right)$$



But I could have also read it directly from the plot:
3 continuous cycles in 20 samples $\therefore f_{\omega_0} = \frac{3}{20}$ cycles/sample

Periodicity of sampled exponentials

Discrete-time signal $x[n]$ periodic with N if: $x[n] = x[n + N]$ for all n

$$\begin{aligned} A e^{j(2\pi f_{\omega_0} n + \theta)} &\stackrel{?}{=} A e^{j(2\pi f_{\omega_0} (n+N) + \theta)} \\ &= A e^{j(2\pi f_{\omega_0} n + 2\pi f_{\omega_0} N + \theta)} \end{aligned}$$

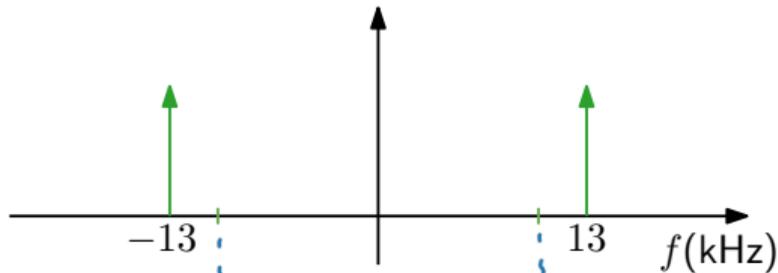
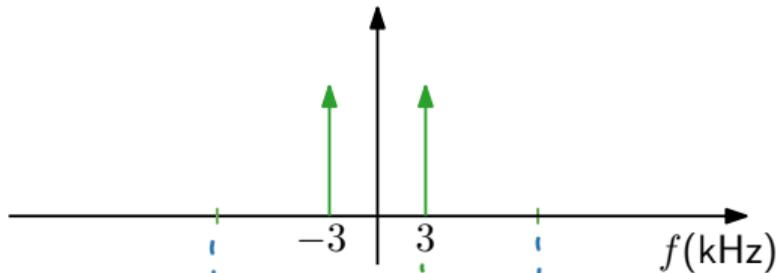
For RHS to equal LHS:

$$2\pi f_{\omega_0} N = 2\pi k$$

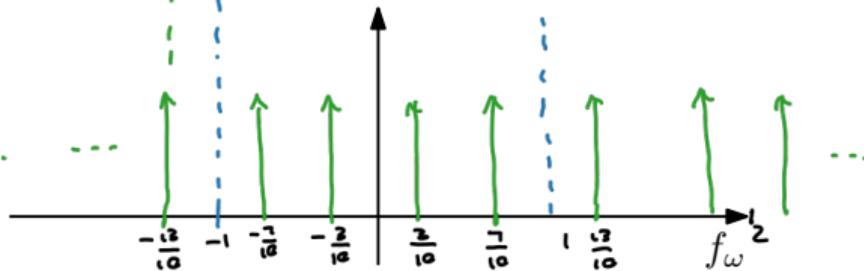
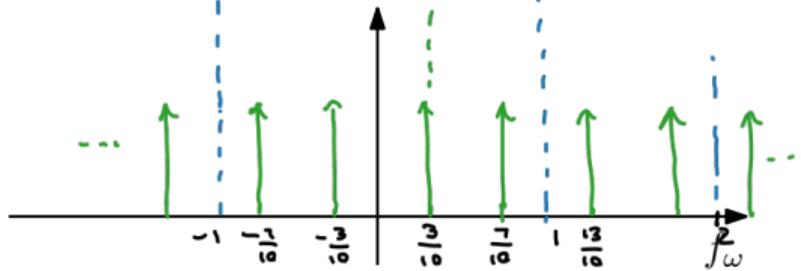
$$\therefore f_{\omega_0} = \frac{k}{N} = \frac{f_0}{f_s}$$

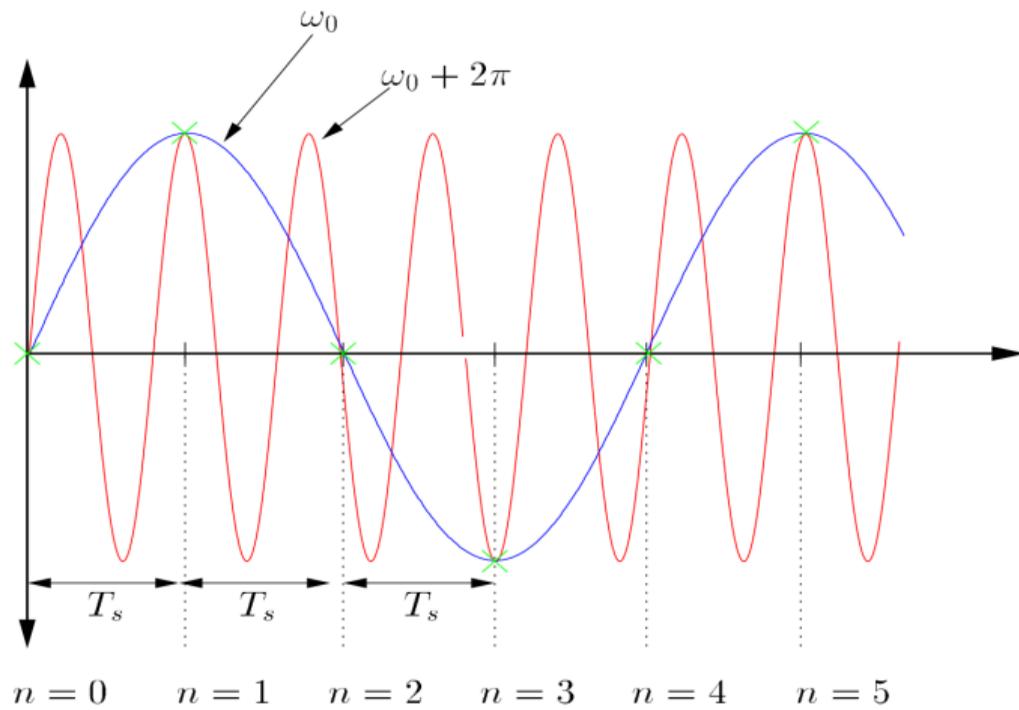
Aliasing of sinusoidal signals

Two sinusoidal signals:

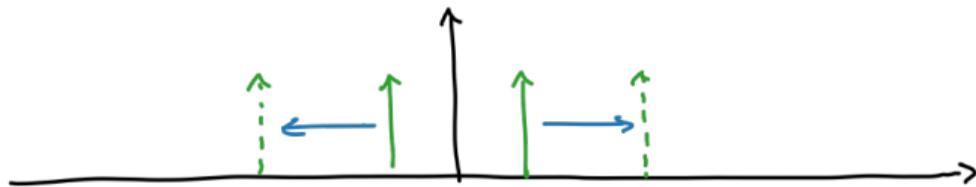


Sample both at $f_s = 10$ kHz:

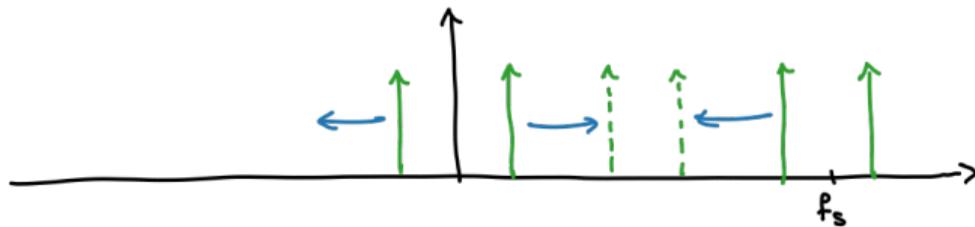




Sinusoid:



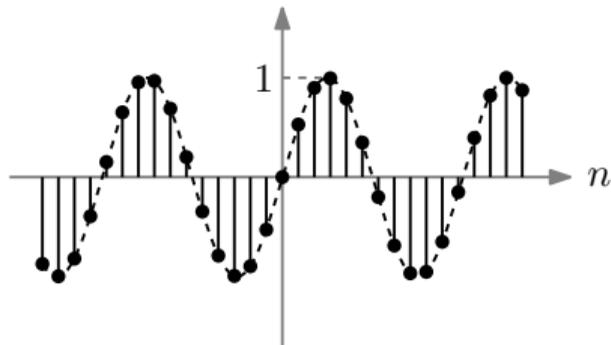
Sampled sinusoid:



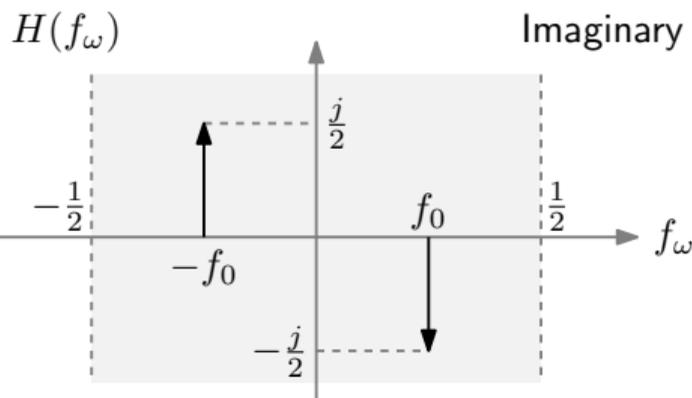
DTFT pairs

Discrete-time domain

$$h[n] = \sin(2\pi f_0 n)$$

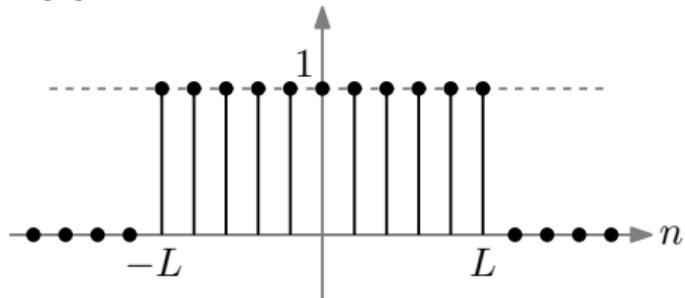


Frequency domain

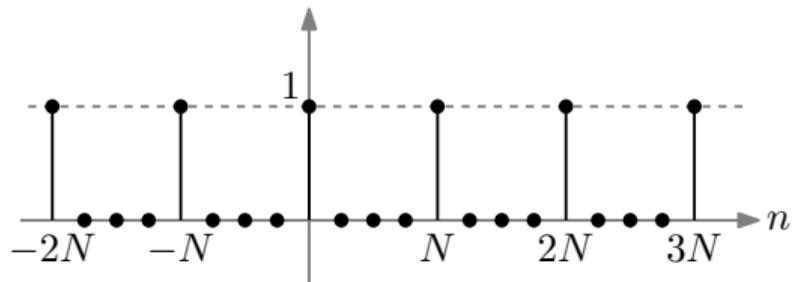


Discrete-time domain

Frequency domain

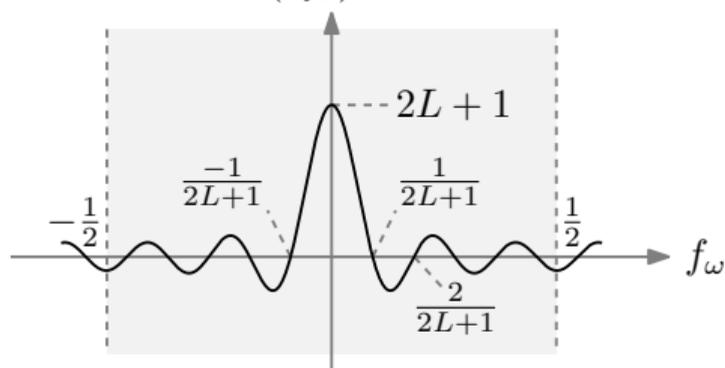
 $h[n]$ 

$$h[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

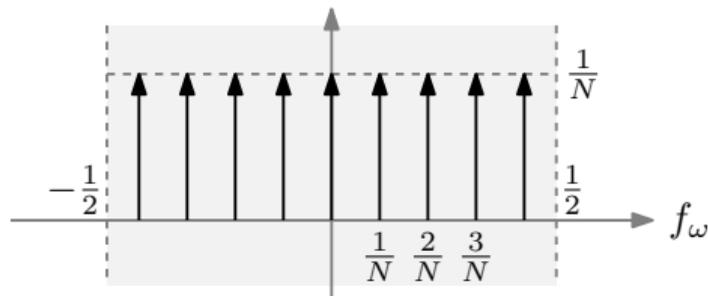


$$H(f_\omega) = \frac{\sin(\pi(2L+1)f_\omega)}{\sin(\pi f_\omega)}$$

Real

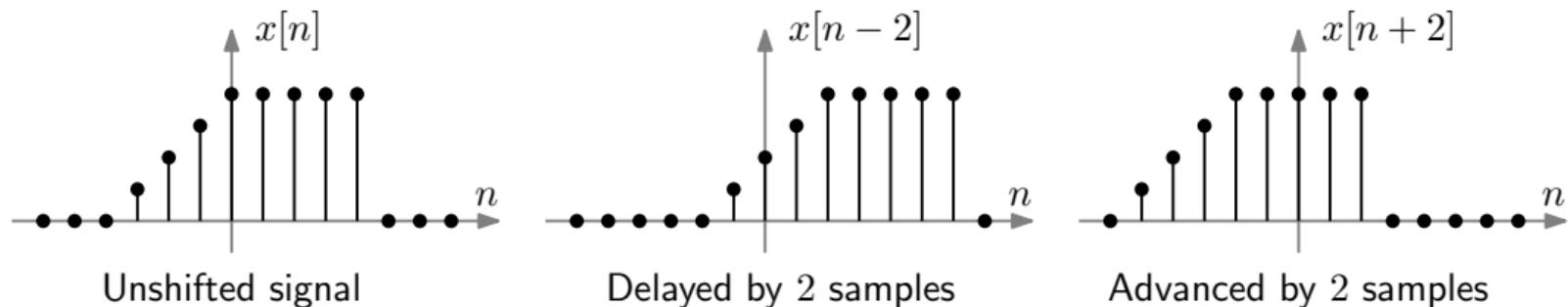


$$H(f_\omega) = \frac{1}{N} \sum_{k=-\infty}^{\infty} \delta\left(f_\omega - \frac{k}{N}\right)$$

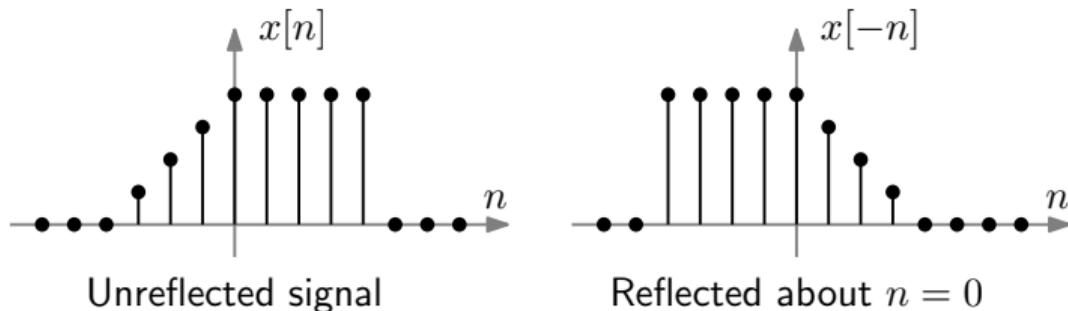


Operations on discrete-time signals

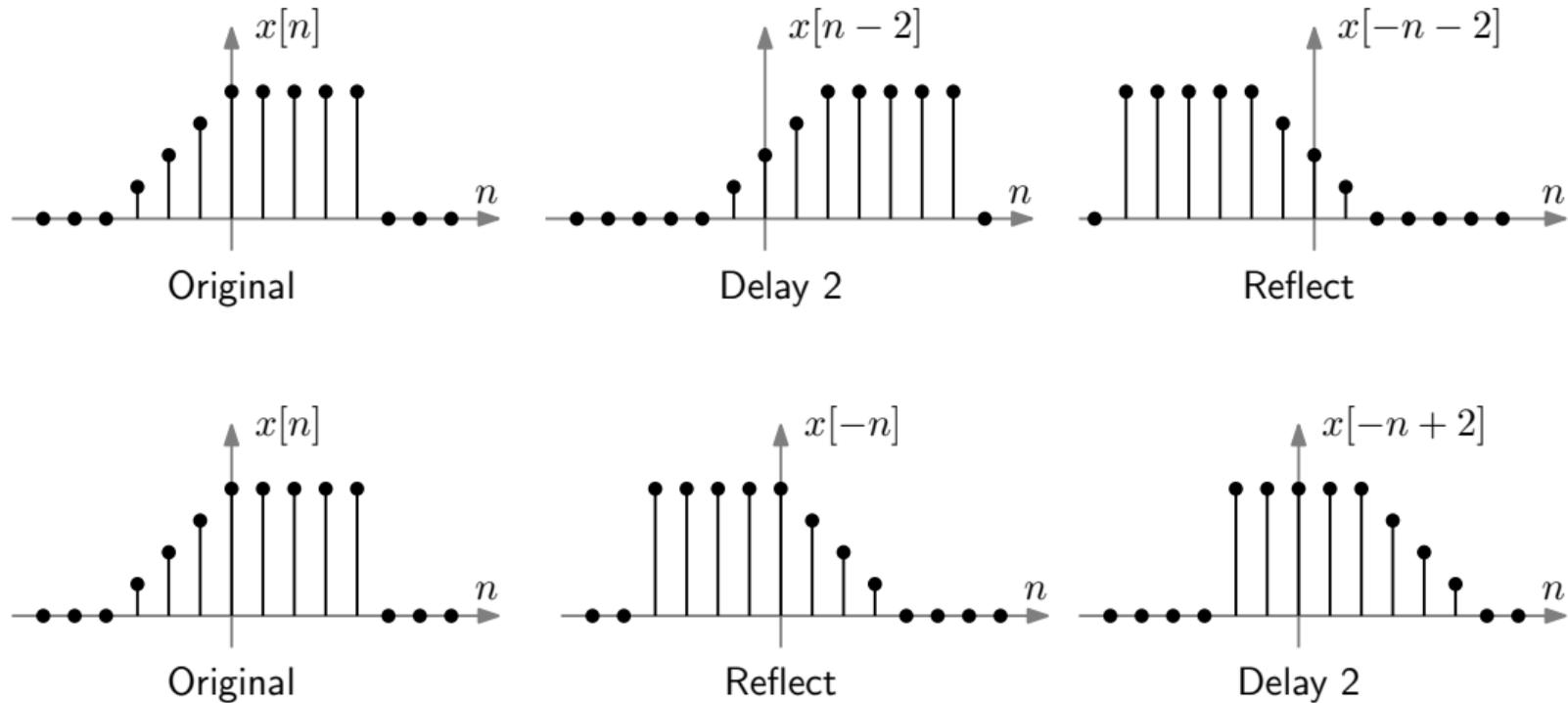
Time shift:



Reflection:



Time-shifting and reflection are not commutative:



Properties of the DTFT

- Linearity:

$$\mathcal{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(\omega) + \beta X_2(\omega)$$

- Time shift:

$$\mathcal{F}\{x[n - k]\} = e^{-j\omega k} X(\omega)$$

- Time reversal and frequency reversal:

$$\mathcal{F}\{x[-n]\} = X(-\omega)$$

- Convolution:

$$\mathcal{F}\{x_1[n] * x_2[n]\} = X_1(\omega) \cdot X_2(\omega)$$

- Windowing:

$$\mathcal{F}\{x_1[n] \cdot x_2[n]\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \cdot X_2(\omega - \lambda) d\lambda$$