Introduction to discrete-time filters

Ideal and elementary filters

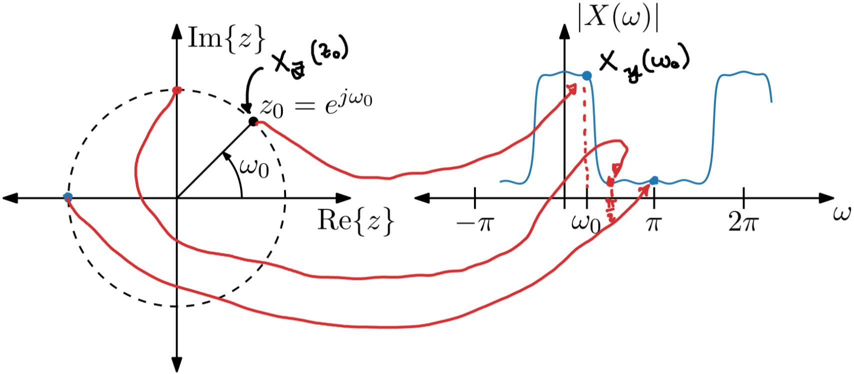
Herman Kamper

Recap: z-transform and DTFT

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$X_y(\omega) = X_z(z)\Big|_{z=e^{j\omega}} = X_z(e^{j\omega})$$



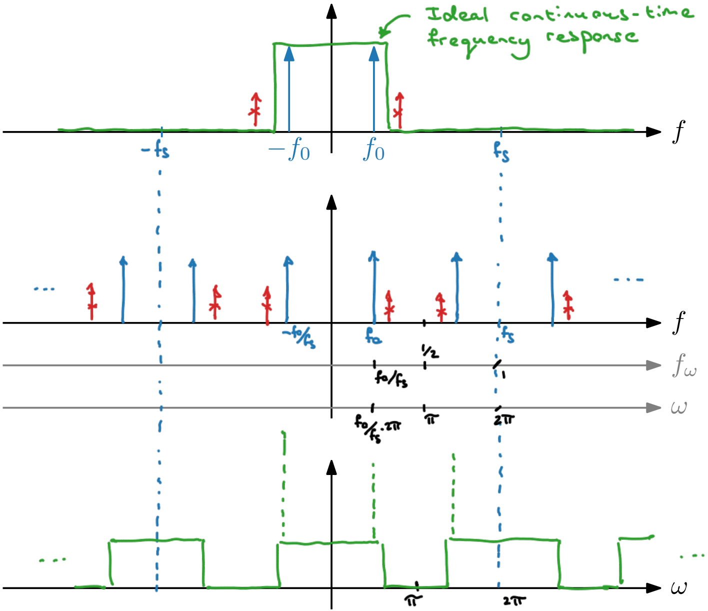
Continuous cosine:

$x(t) = \cos(2\pi f_0 t)$

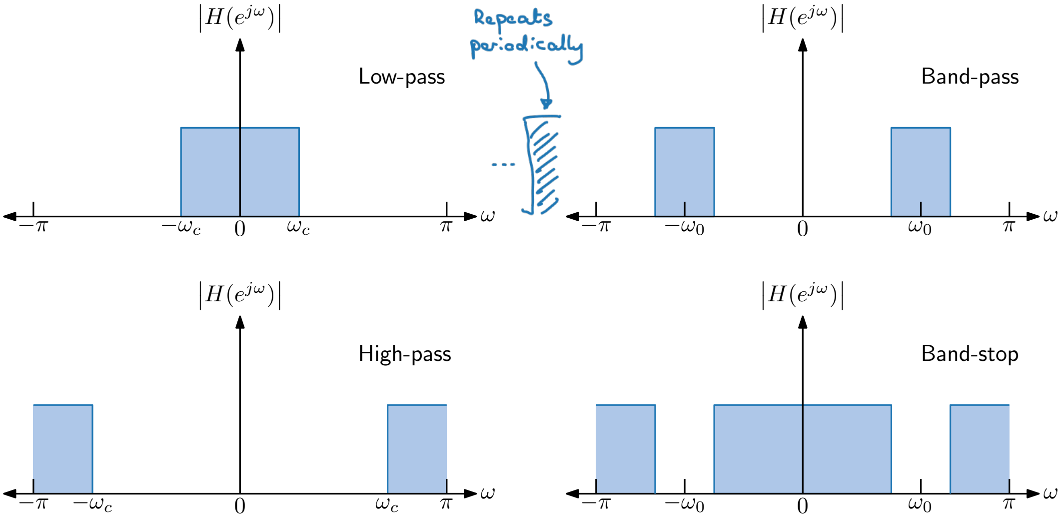
Sample $x(t)$ to get $x[n]$

Design filter to let in everything below

$f_{\omega_0} = \frac{f_0}{f_s}$ (i.e. a LPF)

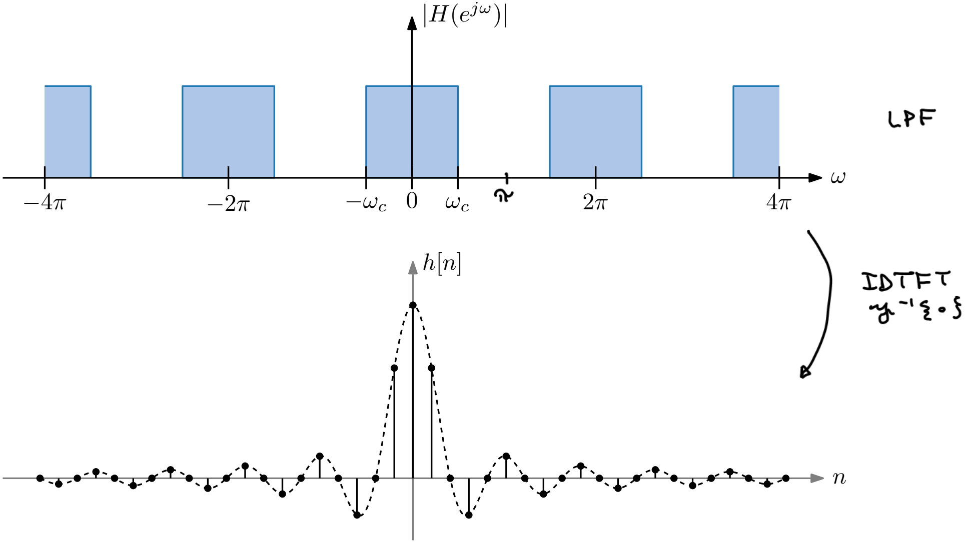


IDEAL FILTERS



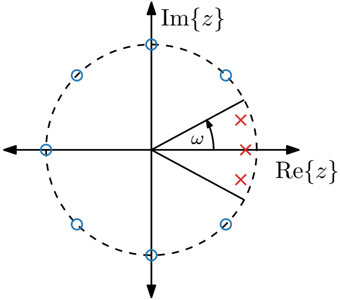
We often only draw the interval from $-\pi$ to π (why?)

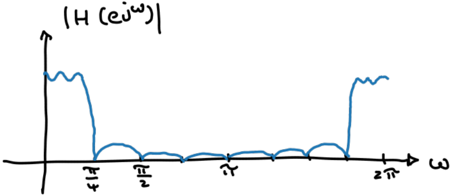
Ideal filters are unrealisable



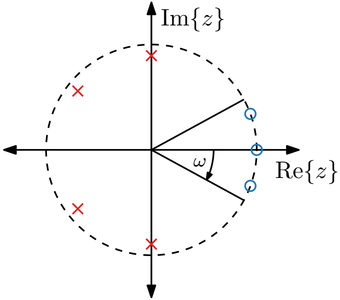
Elementary filters

LPF (1) BSF (2) HPF (3) BPF (4)





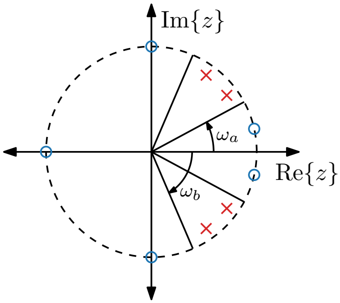
LPF



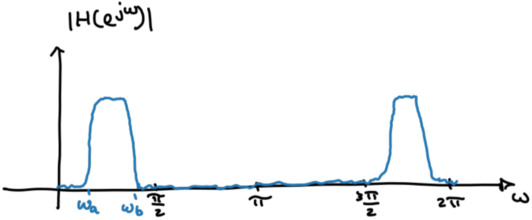
HPF

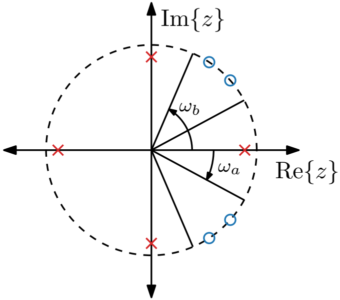
Elementary filters

LPF (1)   BSF (2)   HPF (3)   BPF (4)



BPF



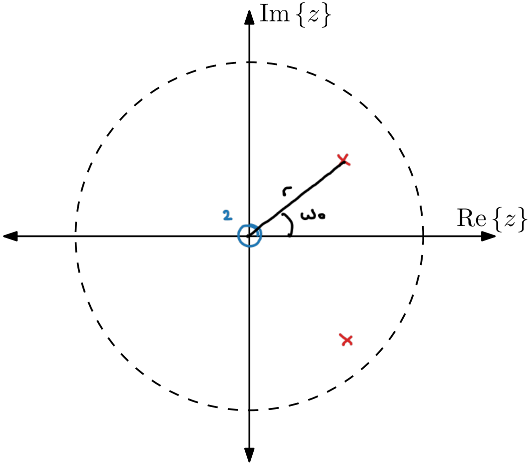


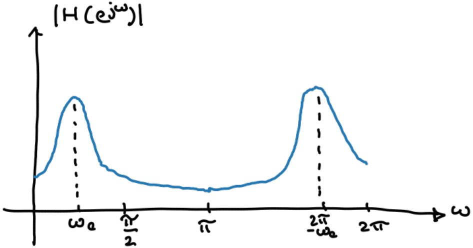
BSF

Digital resonator: An elementary BPF

$$H(z) = \frac{b_0}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} = \frac{b_0}{(1-\underbrace{re^{j\omega_0}z^{-1}}_{\frac{re^{j\omega_0}}{z}})(1-re^{-j\omega_0}z^{-1})}$$

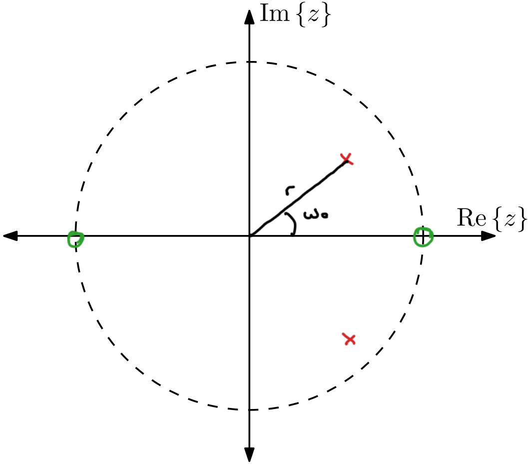
$\therefore\ p_{1,2} = re^{\pm j\omega_0}$

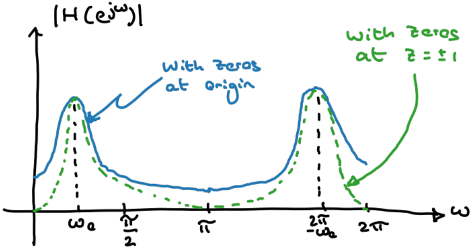


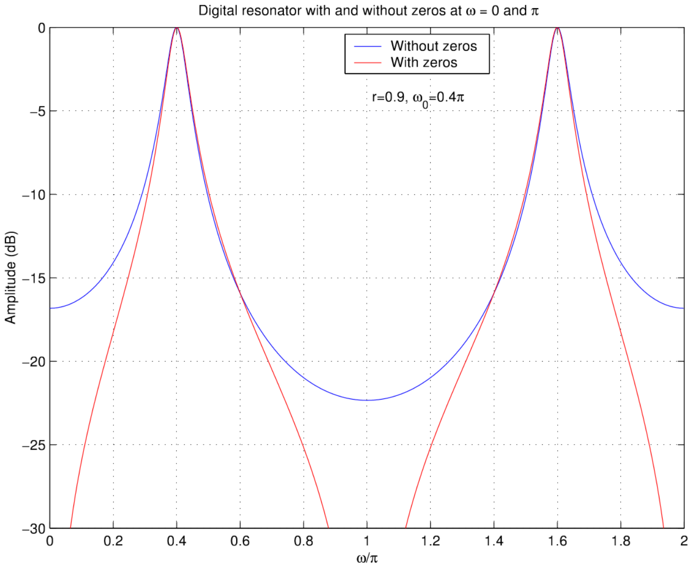


Can deepen nulls by introducing zeros at $z = \pm 1$:

$$H(z) = \frac{b_0(1 - z^{-2})}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}} = \frac{b_0(1 - z^{-1})(1 + z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$





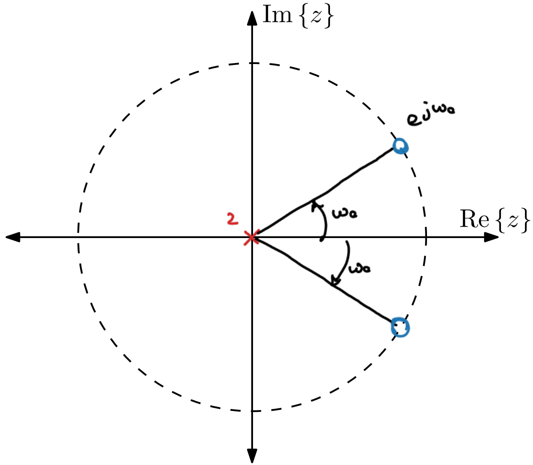


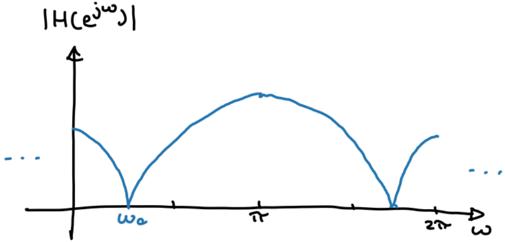
Digital resonator with and without zeros at ω = 0 and π
Without zeros
With zeros
r=0.9, ω0=0.4π
Amplitude (dB)
0
−5
−10
−15
−20
−25
−30
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω/π

Notch filter: An elementary BSF

$$H(z) = b_0(1-(2\cos\omega_0)z^{-1}+z^{-2}) = b_0\left(1-\underbrace{e^{j\omega_0}z^{-1}}_{\frac{e^{j\omega_0}}{z}}\right)\left(1-e^{-j\omega_0}z^{-1}\right)$$

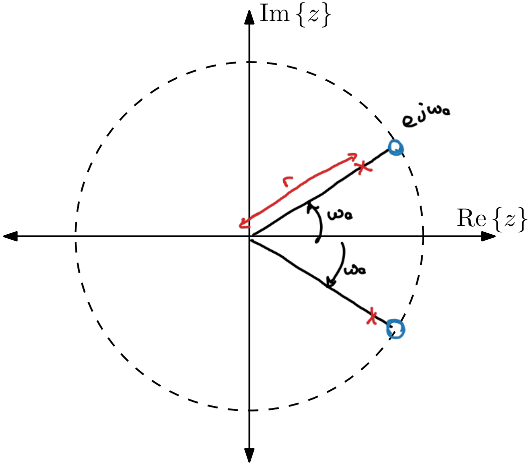
$\therefore \; z_{1,2} = e^{\pm j\omega_0}$

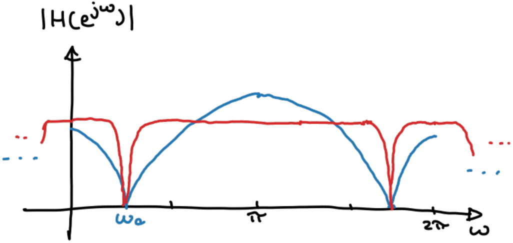


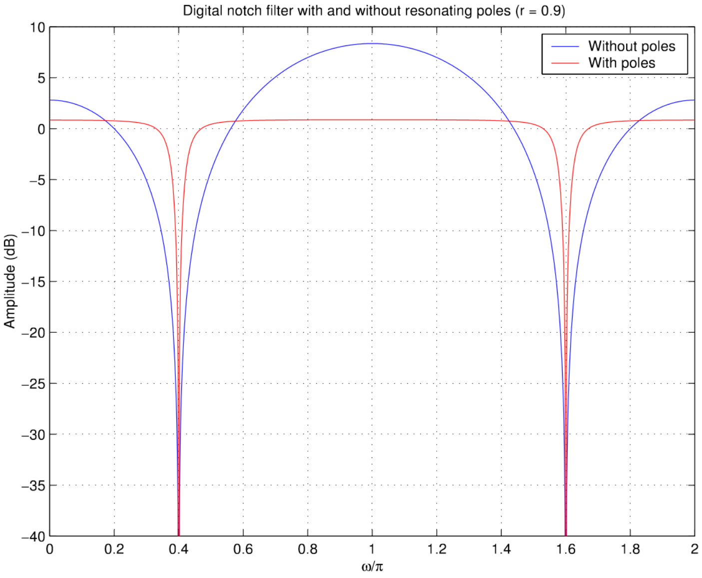


Bandwidth of notches can be reduced by placing a pole at the same frequency close to unit circle:

$$H(z) = b_0 \frac{1 - (2\cos\omega_0)z^{-1} + z^{-2}}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}} = b_0 \frac{\left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)}{\left(1 - re^{j\omega_0} z^{-1}\right)\left(1 - re^{-j\omega_0} z^{-1}\right)}$$

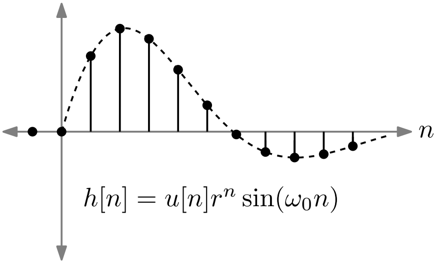





Digital notch filter with and without resonating poles (r = 0.9)
Without poles
With poles
Amplitude (dB)
10
5
0
−5
−10
−15
−20
−25
−30
−35
−40
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω/π

Why not put poles as close as possible to unit circle?

- The derivation that $y[n] = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\omega n + \angle H(e^{j\omega})}$ is for steady state
- This does not take transient effects into account
- What happens below when r gets close to 1?



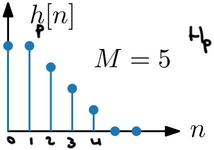
$$H(z) = \frac{(r\sin\omega_0)z^{-1}}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

$$= \frac{(r\sin\omega_0)z^{-1}}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

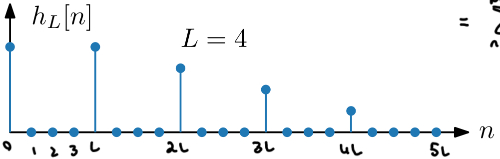
$$|z| > |r|$$

Poles near the unit circle correspond to lightly damped modes, so the system "rings" for a long time after excitation.

Comb filter



$$H_p(e^{j\omega}) = \sum_{n=0}^{M-1} h_p[n] \cdot e^{-j\omega n} \quad \dots \quad (1)$$



$$H_L(z) = \sum_{n=0}^{ML-1} h_L[n] \cdot z^{-n}$$

$$= h_L[0] \cdot z^{-0} + \cancel{h_L[1]} \cdot z^{-1} + \cancel{h_L[2]}\, z^{-2} + \cancel{h_L[3]}\, z^{-3} + h_L[L] \cdot z^{-L}$$

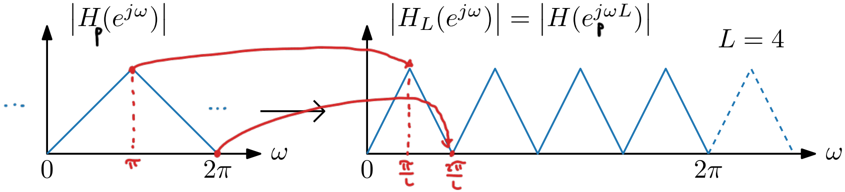
$$+ 0 + \dots + h_L[2L] \cdot z^{-2L} + \dots + h_L[3L]\, z^{-3L} + \dots + h_L[4L] \cdot z^{-4L}$$

$$= h_p[0] \cdot z^{-0} + h_p[1]\, z^{-L} + h_p[2] \cdot z^{-2L} + h_p[3]\, z^{-3L} + h_p[4]\, z^{-4L}$$

$$= \sum_{n=0}^{M-1} h_p[n]\, z^{-nL}$$

$$H_L(e^{j\omega}) = \sum_{n=0}^{M-1} h_p[n] \cdot e^{-j\omega nL}$$

$$= H_p(e^{j\omega L}) \quad (1)$$



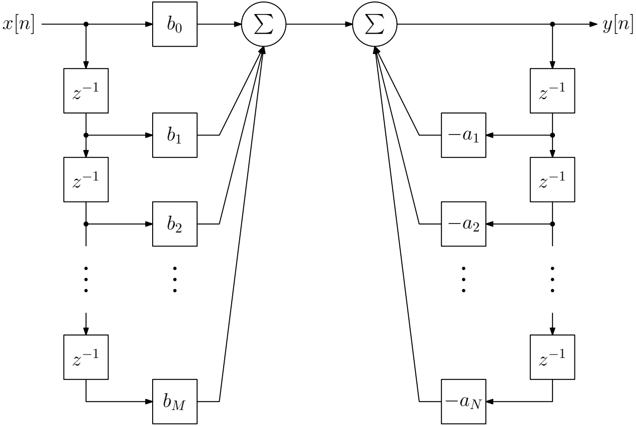
It is like going from $x(t)$ to $y(t) = x(at)$

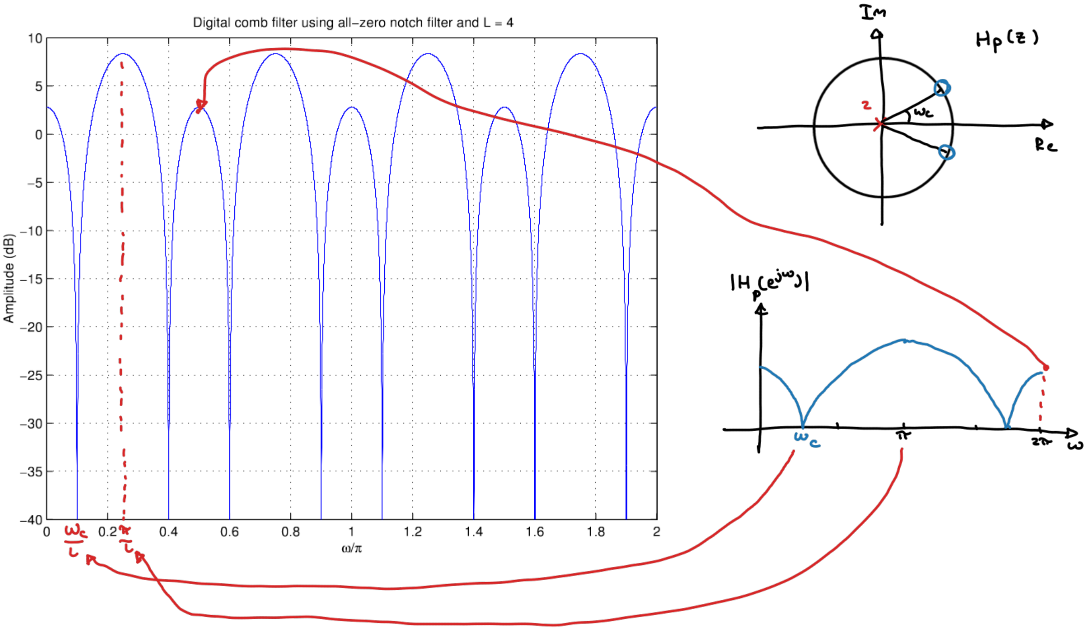
Where have you seen this before?

Comb filter

$b = [1\ ,\ 0\ ,\ -1]$ $\xrightarrow{L=4}$ $b_L = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1]$

$z^{-1} \rightarrow z^{-L}$





Digital comb filter using all−zero notch filter and L = 4
Amplitude (dB)
ω/π
$\frac{\omega_c}{L}$
$\frac{\pi}{L}$
Im
Re
$H_p(z)$
2
ω_c
$|H_p(e^{j\omega})|$
ω_c
π
2π
ω

Why don't we just use the FFT to filter?

- Take the FFT of a signal
- Zero out the components we do not want
- Take the IFFT