Energy and power of discrete signals

Herman Kamper

Energy and power signals

Continuous signals:

$$E = \int_{\infty}^{\infty} v(t)i(t)\,\mathrm{d}t$$

$1\,\Omega$

$$= \int_{\infty}^{\infty} v^2(t)\,\mathrm{d}t$$

$$P = \lim_{t\to\infty} \frac{1}{2T} \int_{-T}^{T} v^2(t)\,\mathrm{d}t$$

Discrete signals:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

If periodic with period N:

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- If E finite $\Rightarrow P = 0$ — Discrete energy signal
- If P finite $\Rightarrow E = \infty$ — Discrete power signal

Parseval's theorem for discrete energy signals

Signal $x_1[n]$ has spectrum $\overset{\text{DTFT}}{X_1(\omega)}$ and $x_2[n]$ has spectrum $\overset{\text{DTFT}}{X_2(\omega)}$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)\cdot X_2^*(\omega)\cdot d\omega$$

DTFT

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\omega n}\right] X_2^*(\omega)\cdot d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\,\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)\cdot e^{-j\omega n}\cdot d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(-\omega')\, e^{j\omega' n}\, d\omega'\right]$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\cdot x_2^*[n]$$

$\omega' = -\omega$

$d\omega' = -d\omega$

ω	ω'
π	$-\pi$
$-\pi$	π

Set $x_1[n] = x_2[n] = x[n]$:

$$\sum_{n=-\infty}^{\infty} x[n]\cdot x^*[n]$$

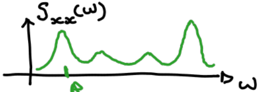
$$= \sum_{n=-\infty}^{\infty} |x[n]|^2 = E_x$$

$$\therefore\ E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot X^*(\omega)\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, \mathrm{d}\omega}$$

Power density spectrum: $\boxed{S_{xx}(\omega) = |X(\omega)|^2}$



a lot of energy due to this frequency

Parseval's theorem for discrete power signals

Periodic $x[n]$ with period N:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\underbrace{x^*[n]}_{\text{IDFT}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n] \left[\frac{1}{N}\sum_{k=0}^{N-1} X^*[N-k]e^{-j\omega k/N}\right]$$

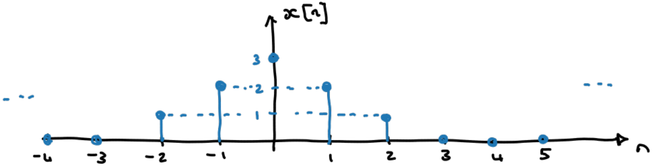
$$= \frac{1}{N}\sum_{n=0}^{N-1} X^*[n] \underbrace{\left[\frac{1}{N}\sum_{k=0}^{N-1} x[k]e^{-j\omega k/N}\right]}_{\text{DFT}} = \frac{1}{N^2}\sum_{n=0}^{N-1} X^*[n]X[n]$$

$$\boxed{P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2}$$

Power density spectrum: $\boxed{S_{xx}[k] = \frac{1}{N}|X[k]|^2}$

Energy signal example

$$x[n] = \begin{cases} 3 - |n| & \text{if } |n| < 4 \\ 0 & \text{otherwise} \end{cases}$$



What are E and P?

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 0^2 + 0^2 + \ldots + 0^2 + 1^2 + 2^2 + 3^2 + 2^2 + 1^2 + 0^2 + 0^2 + \ldots + 0^2$$

$$= 1 + 4 + 9 + 4 + 1$$

$$= 19$$

$$P = 0$$

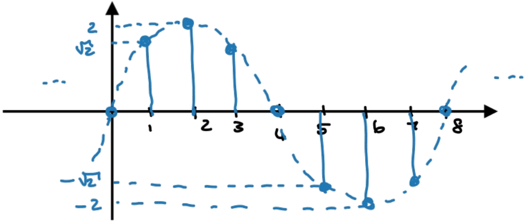
Power signal example

$f_{\omega,0} = \frac{1}{8}$

$x[n] = 2\sin\left(\frac{\pi n}{4}\right) = 2\sin\left(2\pi \cdot \frac{1}{8} n\right)$

(a) What is E? $E = \infty$

(b) Calculate P in two ways.



$$① \; P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{8}(0+2+4+2+0+2+4+2) = \frac{16}{8} = 2$$

$$② \; P = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

$$= \frac{1}{8^2}\left((-8)^2 + 8^2\right)$$

$$= \frac{2 \cdot \cancel{8^2}}{\cancel{8^2}} = 2$$

Take 8-point DFT: