Correlation of discrete energy and power signals

Definitions, calculations and applications

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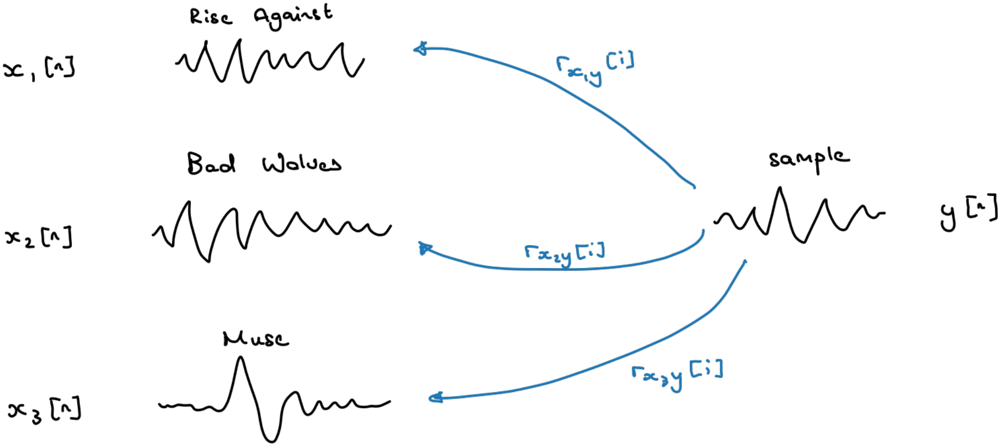
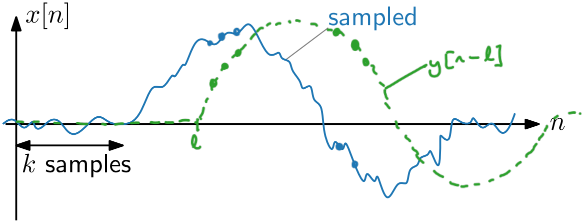
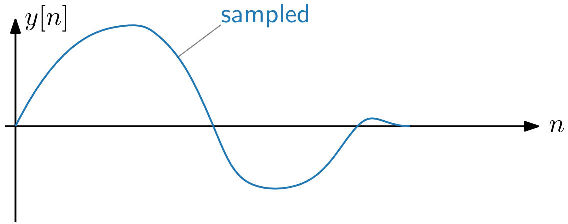
Demo

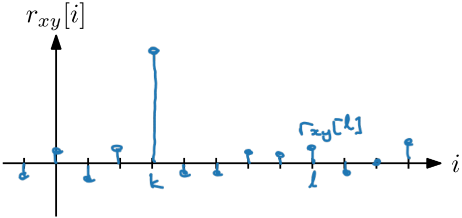

Rise Against
x₁[n]
Bad Wolves
x₂[n]
Muse
x₃[n]
sample
y[n]
r_{x₁y}[i]
r_{x₂y}[i]
r_{x₃y}[i]

Cross-correlation of discrete energy signals

$$r_{xy}[i] = \sum_{n=-\infty}^{\infty} x[n]y[n-i] = \sum_{n=-\infty}^{\infty} x[n+i]y[n]$$







Properties of cross-correlation of energy signals

Cross-correlation is like convolution, but without reflection:

$$x[i] * y[i] = \sum_{n=-\infty}^{\infty} x[n]y[i-n]$$

$$\Rightarrow x[i] * y[-i] = \sum_{n=-\infty}^{\infty} x[n]y[n-i] = r_{xy}[i]$$

Cross-correlation in frequency domain:

$$r_{xy}[i] = x[i] * y[-i]$$

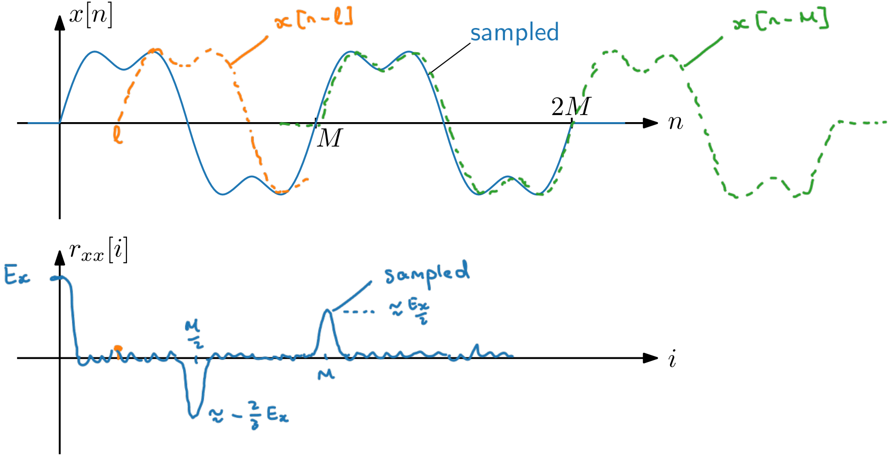
$$\mathcal{F}\left\{r_{xy}[i]\right\} \underset{\text{DTFT}}{=} X(\omega)Y(-\omega)$$

Cross-correlation symmetry:

$$r_{yx}[i] = \sum_{n=-\infty}^{\infty} y[n+i]x[n] = \sum_{n=-\infty}^{\infty} x[n]y[n-(-i)] = r_{xy}[-i]$$

Autocorrelation of discrete energy signals

$$r_{xx}[i] = \sum_{n=-\infty}^{\infty} x[n]x[n-i] = \sum_{n=-\infty}^{\infty} x[n+i]x[n]$$



Properties of autocorrelation of energy signals

Autocorrelation symmetry:

$$r_{xx}[-i] = r_{xx}[i]$$

Autocorrelation and energy:

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x$$

Autocorrelation in frequency domain:

$$\mathcal{F}\left\{r_{xx}[i]\right\} = X(\omega)X(-\omega) = X(\omega)X^*(\omega)$$

$$\Rightarrow \mathcal{F}\left\{r_{xx}[i]\right\} = \underbrace{\left|X(\omega)\right|^2}_{\text{Power spectral density}}$$

Correlation of energy signals using DFT

FFT

Recall that $r_{xy}[i] = x[i] * y[-i]$ when $x[n]$ and $y[n]$ are energy signals

Zero pad $x[n]$ and $y[n]$ appropriately $N \geqslant L+P-1$

Cross-correlation via DFT:

zero-padded (all other signals as well)

$$\tilde{X}[k] = \mathrm{DFT}\left\{\tilde{x}[i]\right\}$$

$$Y[k] = \mathrm{DFT}\left\{y[i]\right\} \Rightarrow \mathrm{DFT}\left\{y[-i]\right\} = Y^*[k]$$

$$\mathrm{DFT}\left\{r_{xy}[i]\right\} = X[k]\, Y^*[k]$$

Bounds

Can proove that:

$$|r_{xx}[i]| \leq r_{xx}[0] = E_x$$

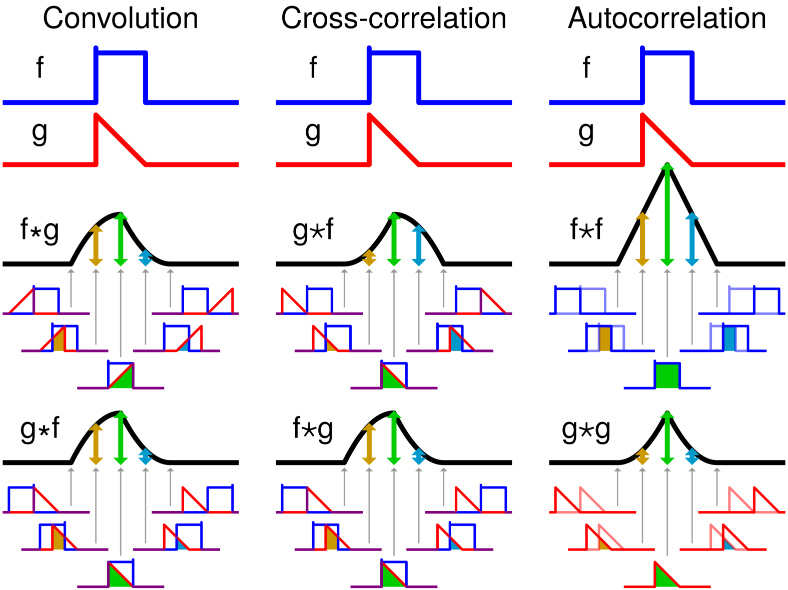
and similarly that:

$$|r_{xy}[i]| \leq \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{E_x E_y}$$

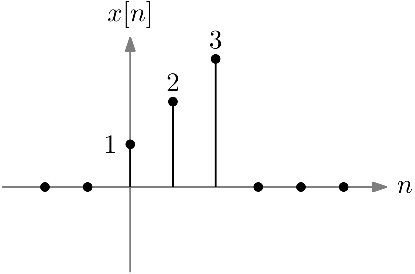
Often scale by upper bounds:

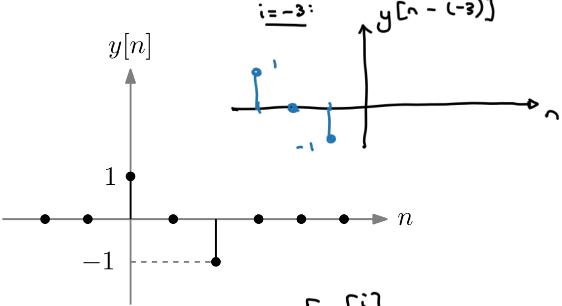
$$\rho_{xx}[i] = \frac{r_{xx}[i]}{E_x} \qquad -1 \leq \rho_{xx}[i] \leq 1$$

$$\rho_{xy}[i] = \frac{r_{xy}[i]}{\sqrt{E_x E_y}} \qquad -1 \leq \rho_{xy}[i] \leq 1$$


Convolution
Cross-correlation
Autocorrelation
f
g
f∗g
g∗f
f
g
g⋆f
f⋆g
f
g
f⋆f
g⋆g

Cross-correlation example





$$r_{xy}[i] = \sum_{n=-\infty}^{\infty} x[n]\cdot y[n-i]$$

$r_{xy}[-3] = 0$

$r_{xy}[-2] = -1$

$r_{xy}[-1] = -2$

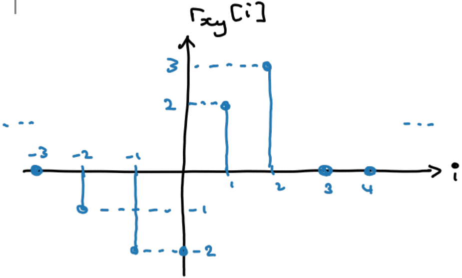
$r_{xy}[0] = 1 + 3(-1) = -2$

$r_{xy}[1] = 2$

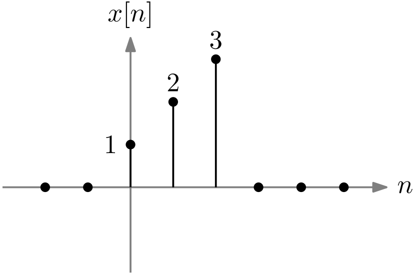
$r_{xy}[2] = 3$

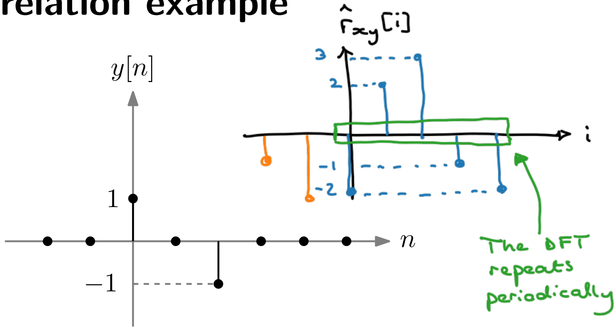
$r_{xy}[3] = 0$

$\vdots$



Cross-correlation example





The DFT repeats periodically

$$N \geq L+P-1 = 3+3-1 = 5$$

$$x[n] = \{\underset{\uparrow}{1} \quad 0 \quad -1\}$$

$$x_{zp}[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3 \quad 0 \quad 0\}$$

$$y_{zp}[n] = \{\underset{\uparrow}{1} \quad 0 \quad -1 \quad 0 \quad 0\}$$

X = np.fft.fft(x_{zp})

Y = np.fft.fft(y_{zp})

$\hat{r}_{xy}$ = np.fft.ifft(X · np.conj(Y))

Correlation of power signals

Cross-correlation of power signals:

$$r_{xy}[i] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]y[n-i]$$

Autocorrelation of power signals:

$$r_{xx}[i] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]x[n-i]$$

Correlation of periodic signals with period N:

$$r_{xy}[i] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]y[n-i]$$

$$r_{xx}[i] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n-i]$$

Autocorrelation and power: $r_{xx}[0] = P_x$

Always

Cross-correlation is like circular convolution, but without reflection:

$$
\begin{aligned}
x[i] \underset{N}{\circledast} y[i] &= \sum_{n=0}^{N-1} x[n]\tilde{y}[i-n] \\
\Rightarrow x[i] \underset{N}{\circledast} y[-i] &= \sum_{n=0}^{N-1} x[n]\tilde{y}[n-i] \\
&= N r_{xy}[i]
\end{aligned}
$$

$x[n]$ and $y[n]$ periodic with period N

Cross-correlation in frequency domain:

$$N \cdot \mathrm{DFT}\left\{r_{xy}[i]\right\} = \mathrm{DFT}\left\{x[i]\right\} \mathrm{DFT}\left\{y[-i]\right\} = X[k]\, Y^*[k]$$

Autocorrelation in frequency domain:

$$N \cdot \mathrm{DFT}\left\{r_{xx}[i]\right\} = X[k]\, X^*[k] = \left|X[k]\right|^2 = N \overbrace{S_{xx}}^{\text{Power spectral density}}[k]$$

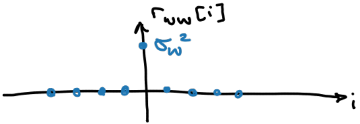
Bounds:

$$|r_{xx}[i]| \leq r_{xx}[0] = P_x$$

$$|r_{xy}[i]| \leq \sqrt{r_{xx}[0]r_{yy}[0]} = \sqrt{P_x P_y}$$

Detecting periodicity using correlation

$$y[n] = \overset{\text{Period } N}{x[n]} + \overset{\text{Noise}}{w[n]}$$



$$r_{yy}[i] = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} y[n] \cdot y[n-i]$$

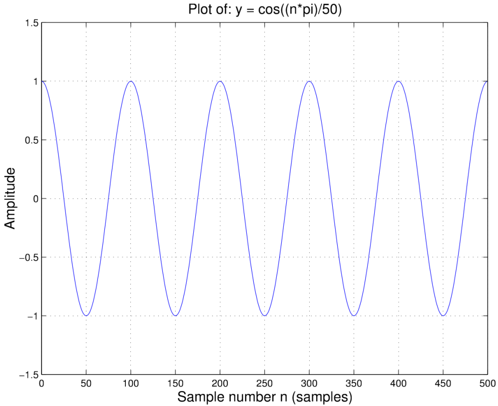
$$= \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \left(x[n] + w[n] \right) \cdot \left(x[n-i] + w[n-i] \right)$$

$$= \lim_{M \to \infty} \frac{1}{2M+1} \left[\sum_{n=-M}^{M} x[n] \cdot x[n-i] + \sum_{n=-M}^{M} x[n] \cdot w[n-i] + \sum_{n=-M}^{M} w[n] \cdot x[n-i] + \sum_{n=-M}^{M} w[n] \cdot w[n-i] \right]$$

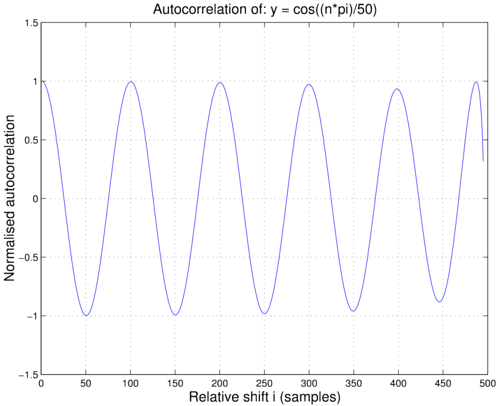
$$= r_{xx}[i] + r_{xw}[i]^{\approx 0} + r_{wx}[i]^{\approx 0} + r_{ww}[i]$$

$\therefore$ When $i \neq 0$: $r_{yy}[i] \approx r_{xx}[i]$

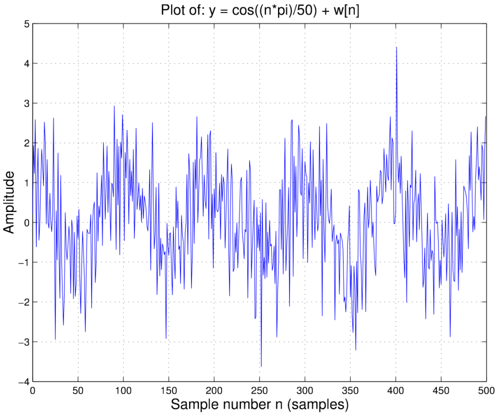
To find period, look for peaks in $r_{yy}[i]$

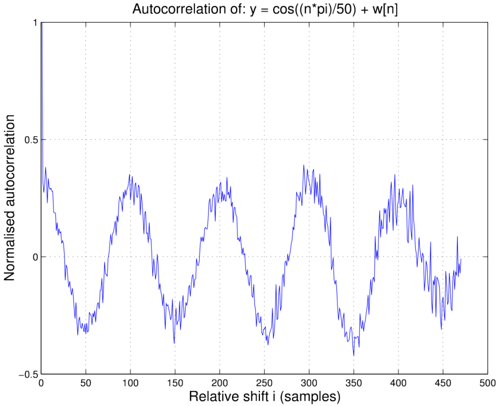


Plot of: y = cos((n*pi)/50)



Autocorrelation of: y = cos((n*pi)/50)


Plot of: y = cos((n*pi)/50) + w[n]
Amplitude
Sample number n (samples)
5
4
3
2
1
0
−1
−2
−3
−4
0
50
100
150
200
250
300
350
400
450
500


Autocorrelation of: y = cos((n*pi)/50) + w[n]
Normalised autocorrelation
Relative shift i (samples)
1
0.5
0
−0.5
0
50
100
150
200
250
300
350
400
450
500