Discrete Fourier transform (DFT)

Getting samples of the spectrum on a computer

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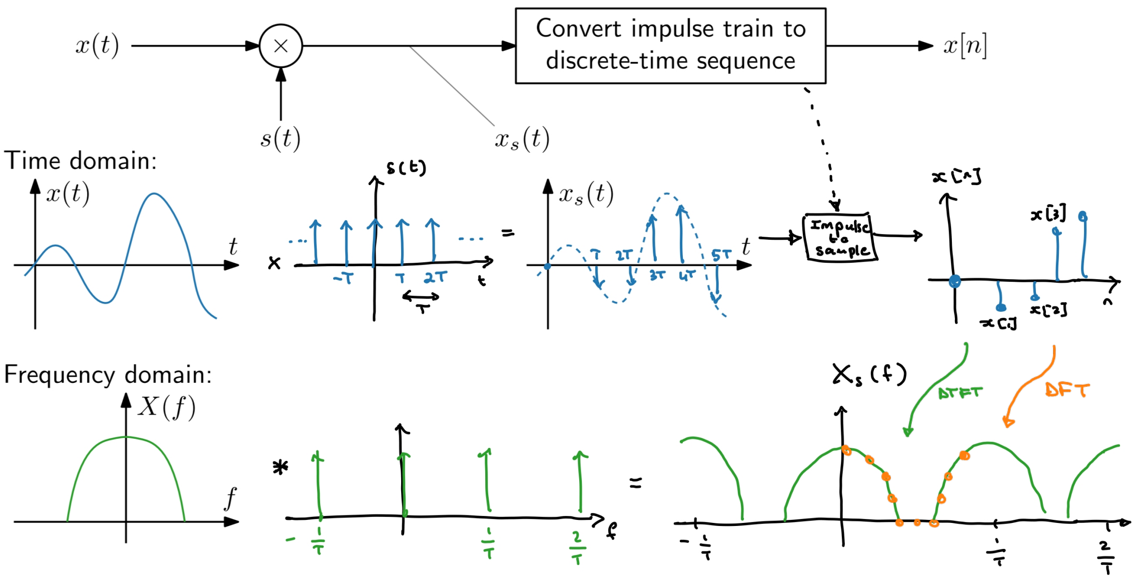
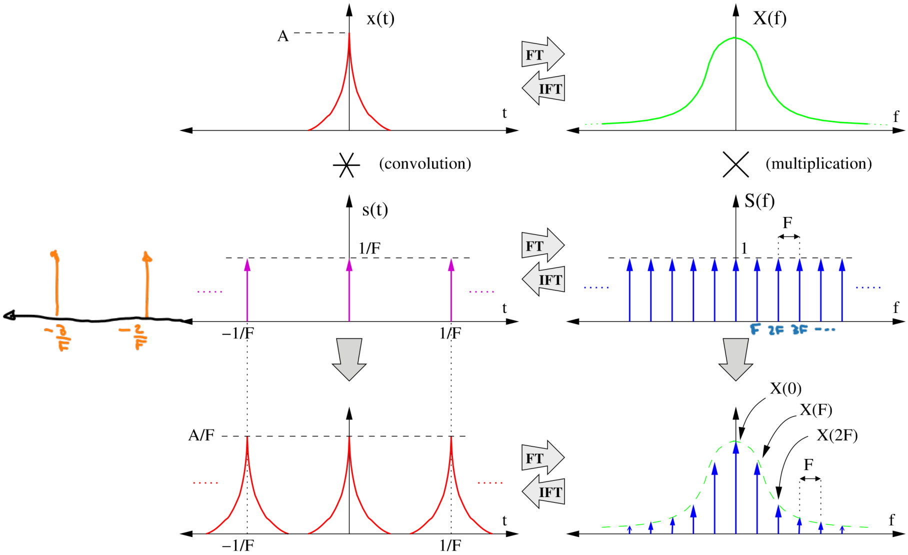
Fourier transform $\longrightarrow$ Discrete-time Fourier transform (DTFT) $\longrightarrow$ $\boxed{\text{Discrete Fourier transform (DFT)}}$

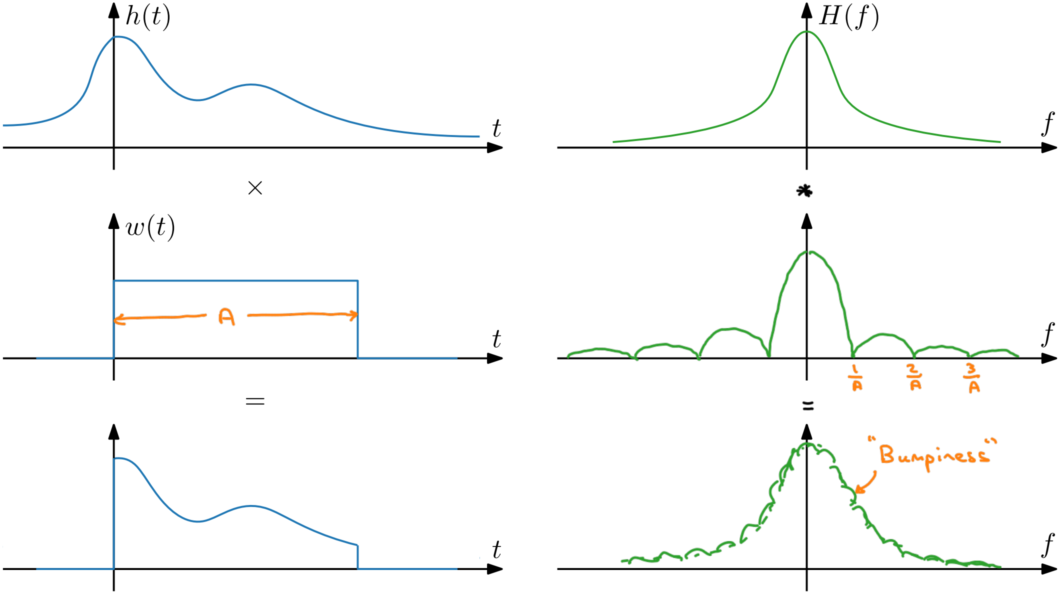
Sampling in the time domain



Sampling in the frequency domain



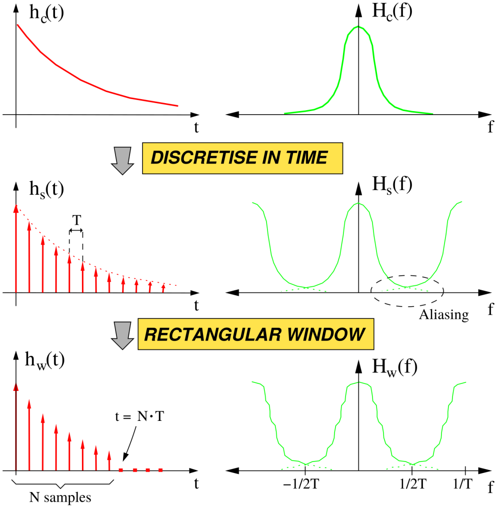
Windowing



Discrete Fourier transform (DFT) steps

1. Sample continuous signal (aliasing)
2. Window the discrete-time signal (bumpiness, ripple)
3. Sample the spectrum (periodic extension in time domain)

Discrete Fourier transform (DFT)



Sampling:

DTFT: $H_s(f) = \sum_{n=-\infty}^{\infty} \overbrace{h(nT)}^{h[n]} e^{-j2\pi f n T}$

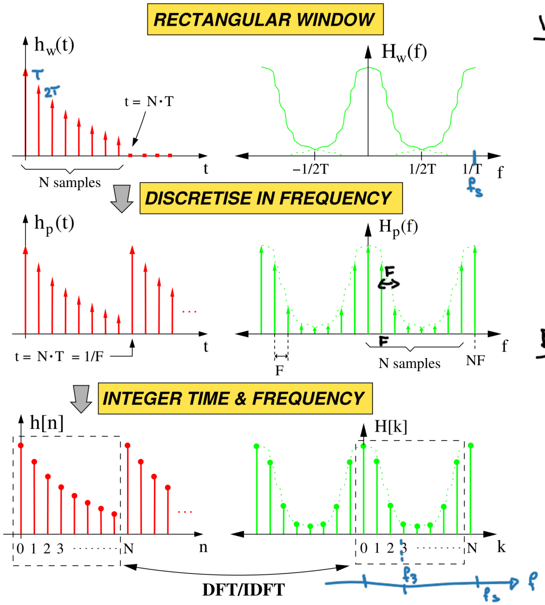
Window:

$$H_w(f) = H_s(f) * W(f)$$

$$= \sum_{n=0}^{N-1} h(nT) e^{-j2\pi f n T}$$

(Assuming rectangular window)

Discrete Fourier transform (DFT)



Window:

$$H_w(f) = H_s(f) * W(f)$$

$$= \sum_{n=0}^{N-1} h(nT)\, e^{-j2\pi f n T}$$

Periodic extension (i.e. sampling spectrum):

$$H_p(f) = H_w(f) \times \sum_{k=-\infty}^{\infty} \delta(f-kF)$$

with $\frac{1}{F} = NT$

Discrete Fourier transform (DFT):

$$h[n] = h(nT)$$

Define e.g. H[2] as strength of impulse at 2F, i.e. $H_p(2F)$. But that is just $H_w(2F)$.

$$H[k] = H_w(kF)$$

$$= \sum_{n=0}^{N-1} h[n]\, e^{-j2\pi kFnT}$$

$$= \sum_{n=0}^{N-1} h[n]\, e^{-j2\pi \frac{k}{N} n}$$

$$kFT = k\frac{1}{N\cancel{T}}\cancel{T} = \frac{k}{N}$$

IDFT:

$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi kn/N}$$

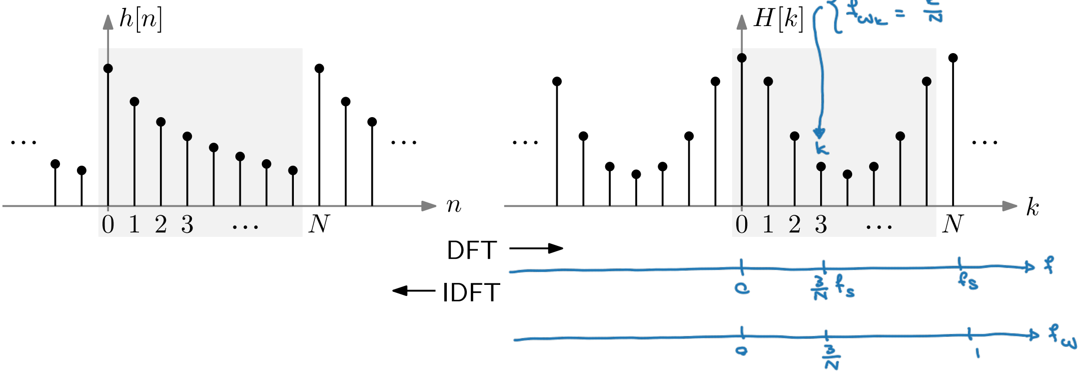
np.fft.ifft

DFT:

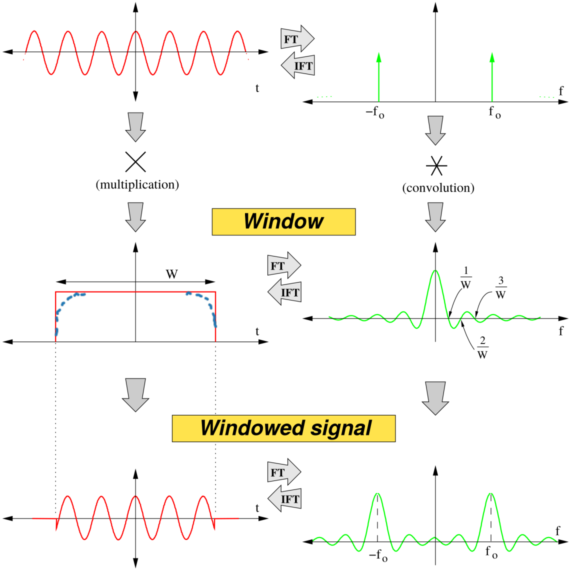
$$H[k] = \sum_{n=0}^{N-1} h[n] e^{-j2\pi kn/N}$$

np.fft.fft

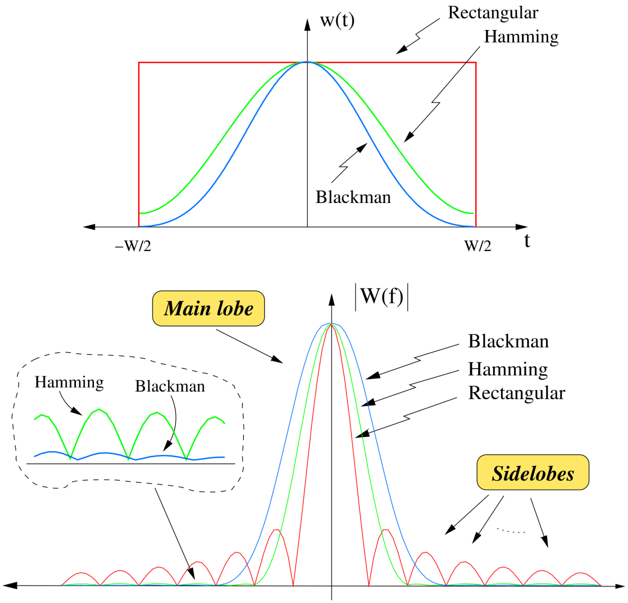
$$\begin{cases} f_k = \frac{k}{N} f_s \\ f_{\omega k} = \frac{k}{N} \end{cases}$$



Windowing



FT
IFT
t
f
−f o
f o
(multiplication)
(convolution)
Window
W
1/W
3/W
2/W
Windowed signal



w(t)
Rectangular
Hamming
Blackman
−W/2
W/2
t
|W(f)|
Main lobe
Blackman
Hamming
Rectangular
Hamming
Blackman
Sidelobes

Inverse discrete Fourier transform (IDFT)

An N-point discrete periodic signal can be expressed as the sum of N complex exponentials with discrete-time frequencies $\omega_k = \frac{2\pi k}{N}$.

If you don't believe me: you've proved it already in the DFT. The discrete $H[k]$ is given by a sum of N complex exponentials. We are now flipping things.

So we know that we should be able to write:

$$h[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}$$

with c_k unkown. The goal is to find all the c_ks.

Multiply the above queation by $e^{-j2\pi nk_1/N}$ on both sides, and take the sum over n:

$$\begin{aligned}
h[n]e^{-j2\pi nk_1/N} &= \left[\sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}\right] e^{-j2\pi nk_1/N} \\
\sum_{n=0}^{N-1} h[n]e^{-j2\pi nk_1/N} &= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j2\pi n(k-k_1)/N} \\
&= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} e^{j2\pi n(k-k_1)/N} \\
&= c_{k_1} N
\end{aligned}$$

That last step looks strange but it comes from

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N & \text{if } k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

which itself comes from the geometric series formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N & \text{if } a = 1 \\ \frac{1-a^N}{1-a} & \text{if } a \neq 1 \end{cases}$$

So we have

$$\sum_{n=0}^{N-1} h[n] e^{-j2\pi nk_1/N} = c_{k_1} N$$

$$\begin{aligned} c_{k_1} &= \frac{1}{N} \sum_{n=0}^{N-1} h[n] e^{-j2\pi nk_1/N} \\ &= \frac{1}{N} H[k_1] \end{aligned}$$

If we plug this back into the equation where we started, we get the IDFT:

$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi nk/N}$$

A different proof just starts with the IDFT result (above) and plugs it into the DFT definition to proove that it is true. We need some of the equations from the previous slide for last step below:

$$\begin{aligned}
H[k] &= \sum_{n=0}^{N-1} h[n] e^{-j2\pi nk/N} \\
&= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{r=0}^{N-1} H[r] e^{j2\pi nr/N} \right] e^{-j2\pi nk/N} \\
&= \frac{1}{N} \sum_{r=0}^{N-1} H[r] \left[\sum_{n=0}^{N-1} e^{j2\pi nr/N} e^{-j2\pi nk/N} \right] \\
&= \frac{1}{N} \sum_{r=0}^{N-1} H[r] \left[\sum_{n=0}^{N-1} e^{j2\pi n(r-k)/N} \right] \\
&= H[k]
\end{aligned}$$