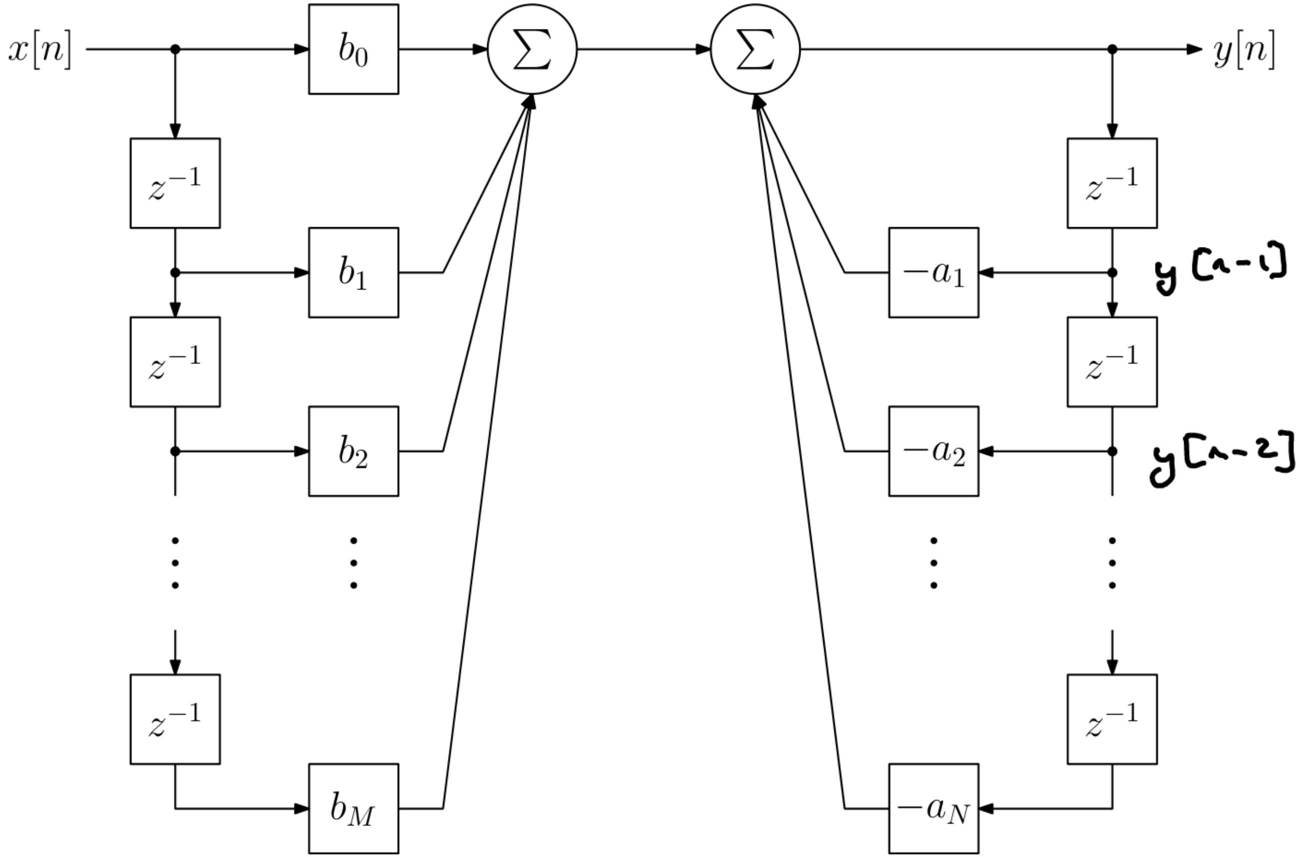
# LTI systems with the z-transform

Herman Kamper




- What do you call a LCCDE system where $N = 0$? FIR
- What can you tell me about the impulse response of system where $N > 0$? IIR
- When is an LTI system BIBO stable? $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

# Three identities we will use today

- Linearity:
$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha \mathcal{Z}\{x[n]\} + \beta \mathcal{Z}\{y[n]\}$$

- Time shift:
$$\mathcal{Z}\{x[n-k]\} = z^{-k}\mathcal{Z}\{x[n]\}$$

- Convolution:
$$\mathcal{Z}\{x[n] * y[n]\} = \mathcal{Z}\{x[n]\} \cdot \mathcal{Z}\{y[n]\}$$

# Transfer function

Linear time-invariant (LTI) system:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad y[n] = x[n] * h[n]$$

$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\{x[n] * h[n]\}$$

$$Y(z) = X(z) \cdot H(z)$$

Transfer function
$H(z) = \mathcal{Z}\{h[n]\}$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

# Transfer functions of LCCDE systems

$$Y(z) = X(z) \cdot H(z)$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)}$$

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$\downarrow \mathcal{Z}\{\cdot\}$$

$$Y(z) = -\sum_{k=1}^{N} a_k \mathcal{Z}\{y[n-k]\} + \sum_{k=0}^{M} b_k \mathcal{Z}\{x[n-k]\}$$

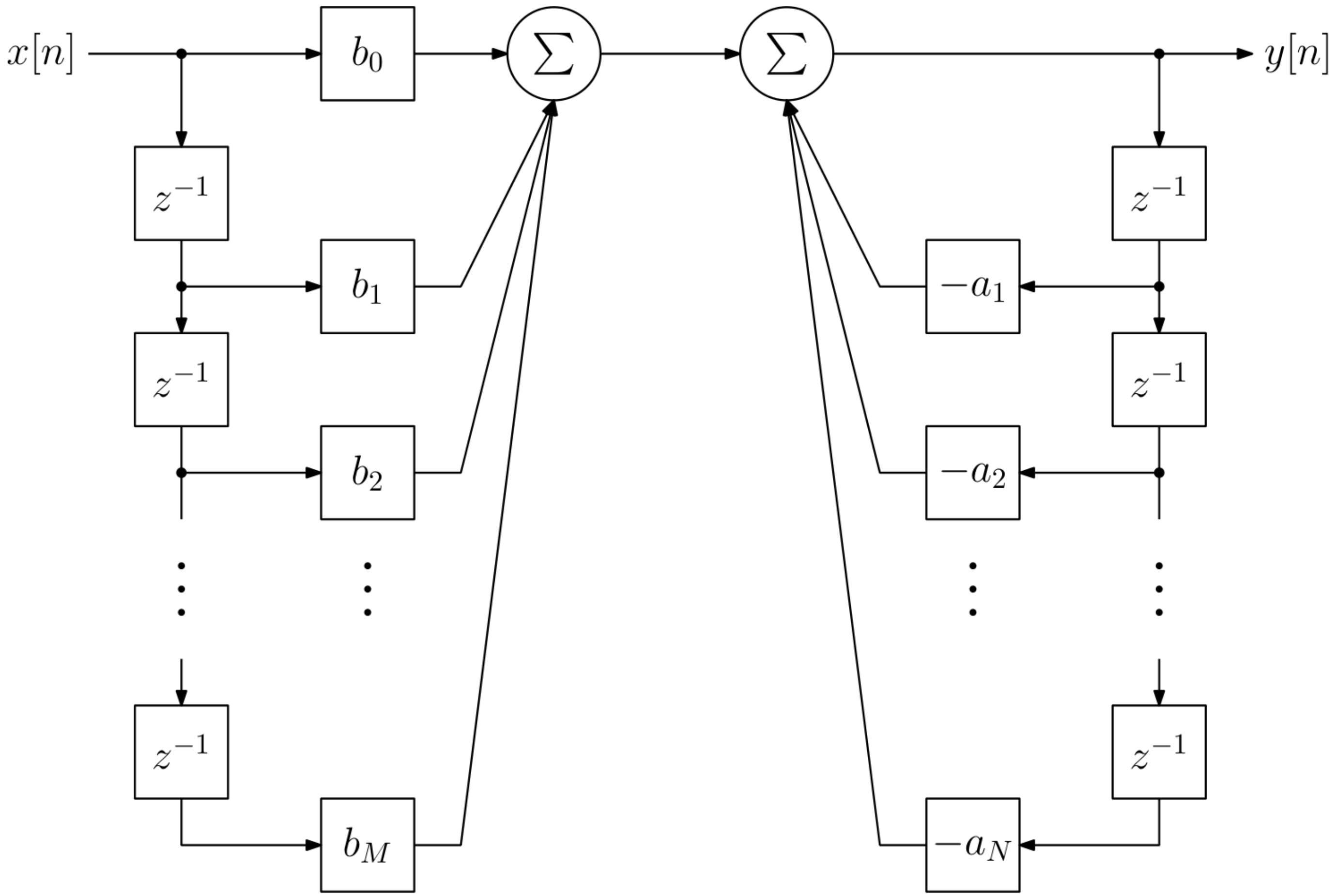
$$= -\sum_{k=1}^{N} a_k \cdot z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z) \cdot \left[\sum_{k=0}^{M} b_k z^{-k}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} \quad \times \underbrace{\frac{z^M}{z^N} \cdot z^{N-M}}_{=1}$$

$$= z^{N-M} \cdot \frac{\sum_{k=0}^{M} b_k \cdot z^{M-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}}$$

$$H(z) = z^{N-M} \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}}$$



# Poles and zeros

LCCDE:

$$H(z) = z^{N-M} \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}} = b_0 z^{N-M} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

$$H(z) = z^{-5} \frac{(z-1)(z-2)}{z-4}$$

$$\frac{1}{z^5}$$

$$= b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

Example:

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Zero: $z = 0$

Pole: $z = a$

LCCDE: $$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

With $N=0$ we have an all-zero system $\Rightarrow$ FIR

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}$$

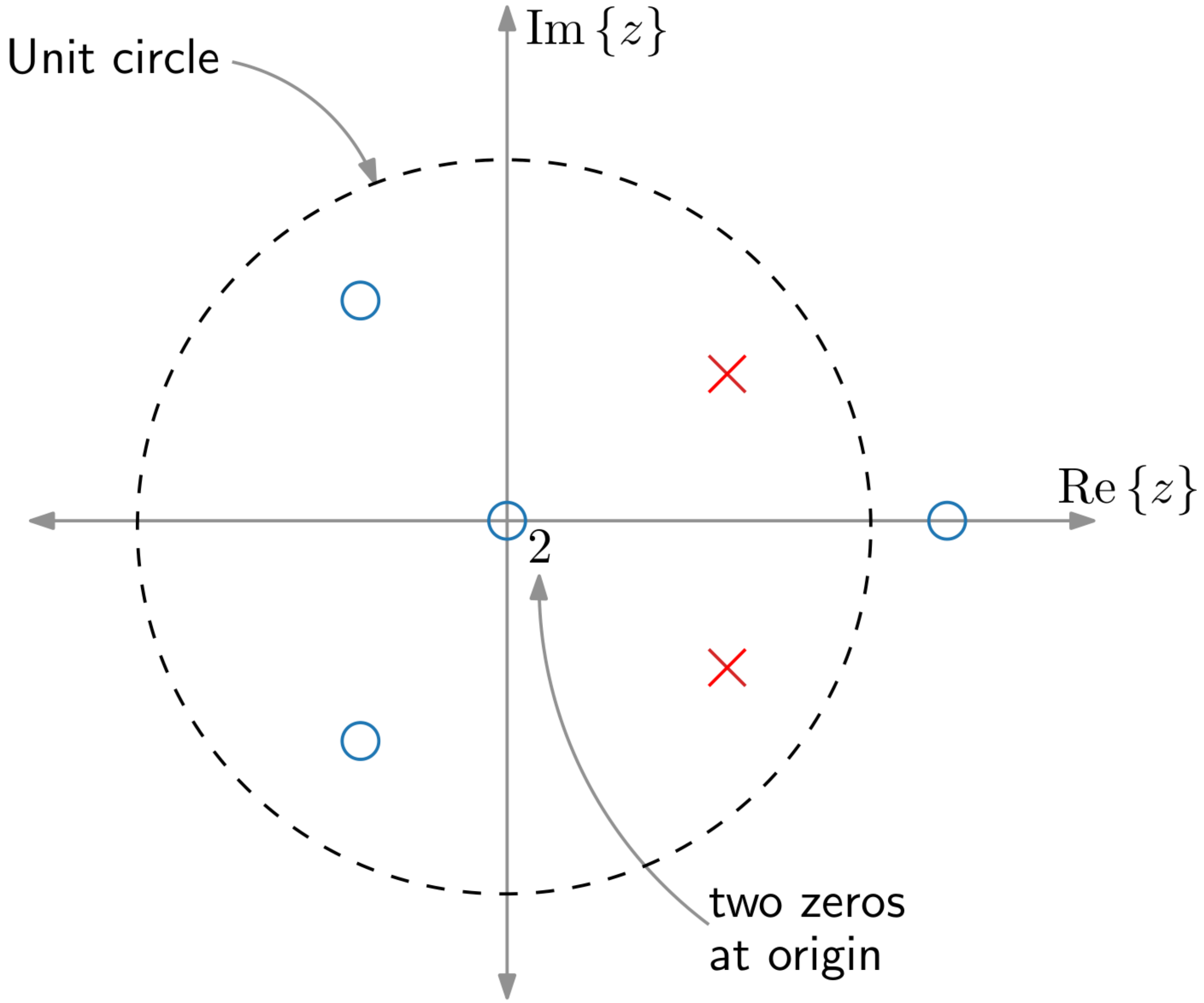
Still have $M$ poles at origin

With $M=0$ we have an all-pole system $\Rightarrow$ IIR

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

$$= \frac{b_0 z^N}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N}$$

Still have $N$ zeros at origin


Unit circle
Im {z}
Re {z}
2
two zeros
at origin

# Causal and anti-causal *signals*

LTI system: $h[n]$

Previous lecture:

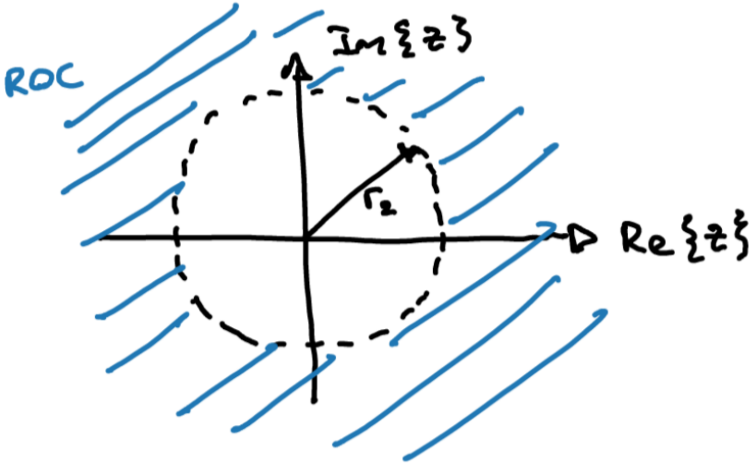
$x[n] = 0$ for all $n < 0$

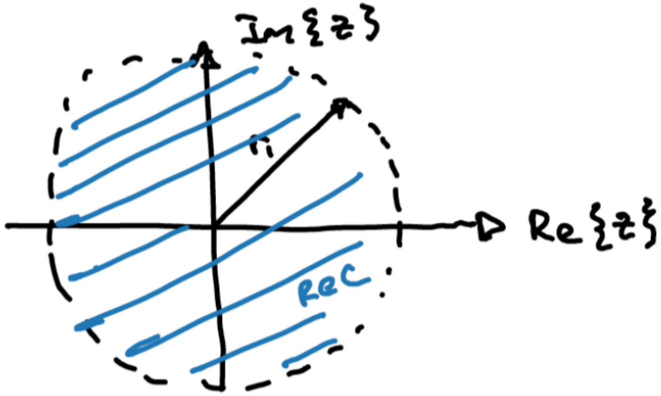
**Causal:**

$$|X(z)| \le \sum_{n=0}^{\infty} \left| x[n] \cdot r^{-n} \right| = \sum_{n=0}^{\infty} \left| x[n] \cdot \frac{1}{r^n} \right|$$

**Anti-causal:**

$$|X(z)| \le \sum_{n=1}^{\infty} \left| x[-n] \cdot r^{n} \right|$$

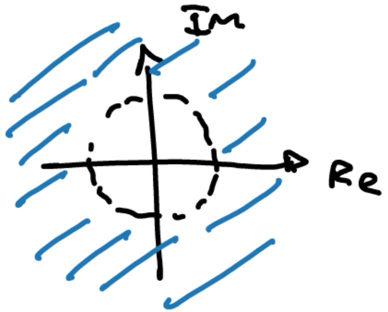


# Stability of causal LTI *systems*

**Causality:**

If the ROC of an LTI system's transfer function $H(z)$ is the exterior of a circle with some radius $r < \infty$, the system is causal.




$h[n] = 0$ for all $n < 0$

**Stability:**

If the ROC of an LTI system's transfer function $H(z)$ includes the unit circle, the system is BIBO stable.

All poles of a stable causal LTI system lie inside the unit circle. (and vice versa)

See the note "Necessity, sufficiency and stability"

BIBO stable: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Inside ROC:

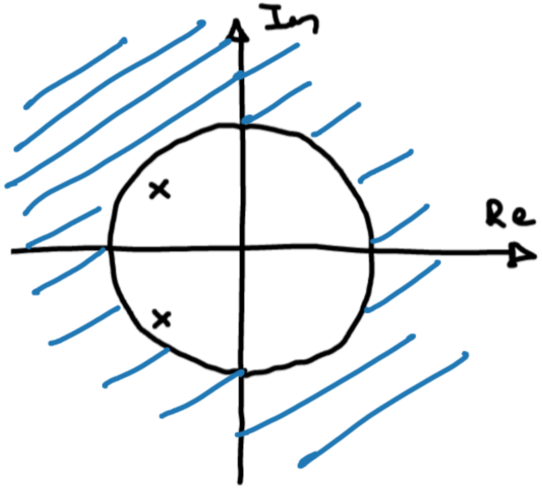
$r = |z|$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n] \cdot r^{-n}| < \infty$$

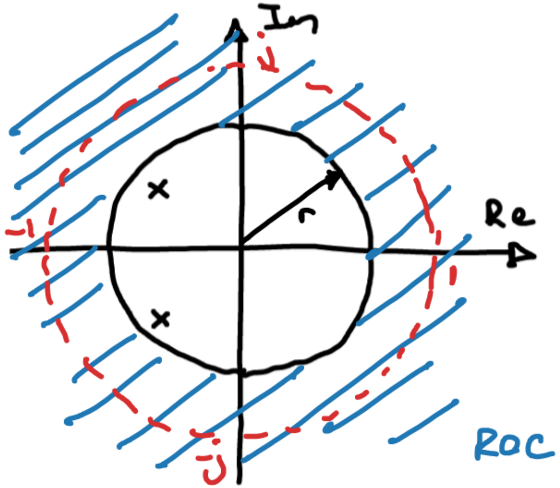
if ROC includes $r = |z| = 1$:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

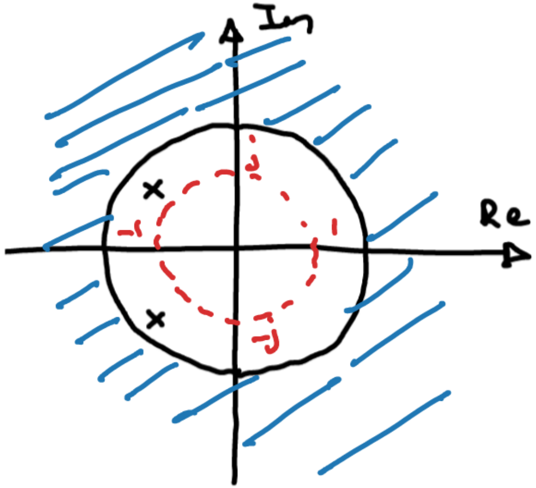
**Always:**




**Stable:**

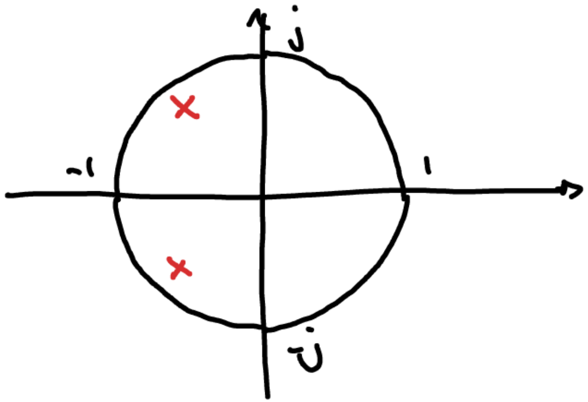



**Unstable:**



BIBO stable $\Rightarrow$ poles inside $r < 1$
(necessary)

poles inside $r < 1 \Rightarrow$ BIBO stable
(sufficient)



# Stability of causal LTI systems