Necessity, sufficiency and stability

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Necessity and sufficiency

Just because something is necessary does not mean it is also sufficient, and vice versa (Wikipedia).

Being a mammal is necessary to be a human, i.e. human $\Rightarrow$ mammal. But it is not sufficient: being a mammal does not mean that you are automatically a human. I.e. it does not follow that mammal $\Rightarrow$ human.

For x to be a rational number it is sufficient that x is real, i.e. rational $\Rightarrow$ real. But it isn't necessary for x to be rational in order to be real, since there are real numbers that are irrational. I.e. it does not follow that real $\Rightarrow$ rational.

BIBO stability

In the lecture Linear time-invariant (LTI) systems we prove that having a impulse response that is absolutely summable is *sufficient* for an LTI system to be BIBO stable:

$$\sum_{n=\infty}^{\infty} |h[n]| < \infty \quad \Rightarrow \quad \text{BIBO}$$

But is it *necessary* for $S_y = \sum_{n=\infty}^{\infty} |h[n]| < \infty$ in order to have BIBO stability? Is it possible to have a case where $S_y = \infty$ but the system is still BIBO stable? To be clear: our proof says nothing about this. But Section 2.3.6 of (Proakis and Manolakis 2007) proves that it is in fact also necessary, i.e.

$$\text{BIBO} \quad \Rightarrow \quad \sum_{n=\infty}^{\infty} |h[n]| < \infty$$

So $S_y < \infty$ is sufficient and necessary. Or, stated differently, an LTI system is BIBO stable *if and only if* $S_y < \infty$. We can write this as

$$\text{BIBO} \quad \Leftrightarrow \quad \sum_{n=\infty}^{\infty} |h[n]| < \infty$$

Stability via the z-transform: Unit circle and region-of-convergence

A causal LTI system is stable *if and and only if* the region-of-convergence (ROC) includes the unit circle, i.e. this is necessary and sufficient. This follows from the BIBO stability result above. If the unit circle is in the ROC, then the impulse response sums to a finite number, which means the system is stable (sufficient). If the unit circle is outside of the ROC, then the impulse response will not sum to a finite number, so the system will be unstable: it is also necessary to have the unit circle in the ROC.

Poles inside the unit circle

In the lecture LTI systems with the z-transform we prove that if a causal LTI system is BIBO stable, then its poles will always be inside the unit circle (necessary). We have not proven that it is also sufficient for for the poles to be inside the unit circle. But it turns out that it is, i.e. if the poles are inside the unit circle, then the causal LTI system is BIBO stable (Sec. 3.5.5, Proakis and Manolakis 2007). The poles need to be strictly inside, not on the unit circle, otherwise we get marginal stability (as discussed in the lecture).

We have not proven the sufficient direction, but it is a intuitive. Per definition, the poles are the values for z where the z-transform goes to infinity. So these should place bounds on the ROC. You can see this in the lecture Introduction to the z-transform for the example of $x[n] = a^n u[n]$ with $0 < a < 1$, where the ROC is $|z| > |a|$ and the pole is at a.

References

J. G. Proakis and D. G. Manolakis, *Digital Signal Processing: Principles, Algorithms and Applications*, 4th ed., 2007.