Summary: Digital signal processing

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Identities

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r} \quad \text{for all } r$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad \text{for } |r| < 1$$

$$\sum_{n=N}^{\infty} r^n = \frac{r^N}{1 - r} \quad \text{for } |r| < 1$$

$$\sum_{n=0}^{\infty} n r^n = \frac{r}{(1 - r)^2} \quad \text{for } |r| < 1$$

Continuous signals

Sinusoidals:

$$\begin{aligned} e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\ \cos(\theta) &= \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta} \\ \sin(\theta) &= \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta} \end{aligned}$$

Even and odd functions:

$$x(t) = x_e(t) + x_o(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

Continuous convolution:

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) \, \mathrm{d}\tau$$

Fourier transform

Fourier transform of $h(t)$:

$$\mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,\mathrm{d}t = H(f)$$

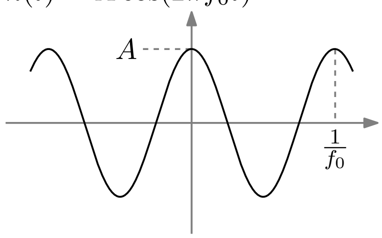
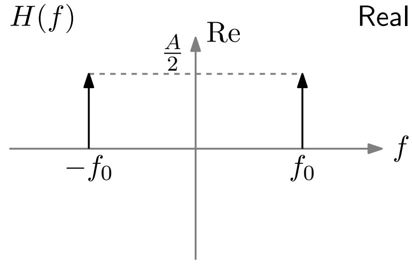
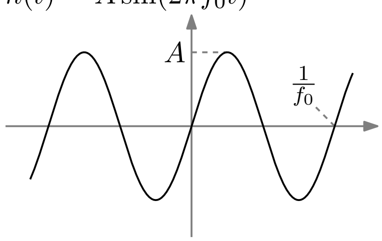
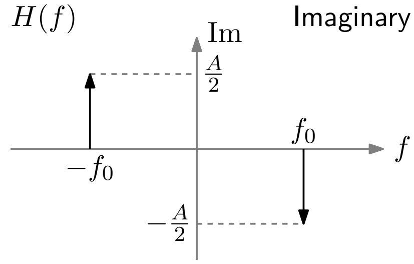
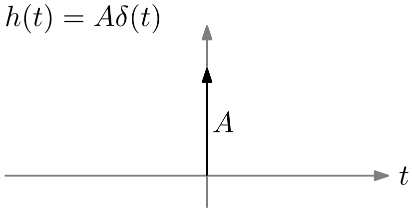
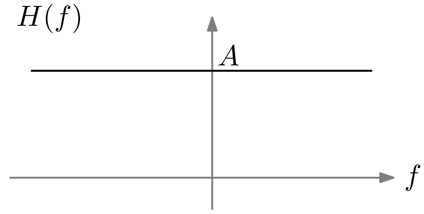
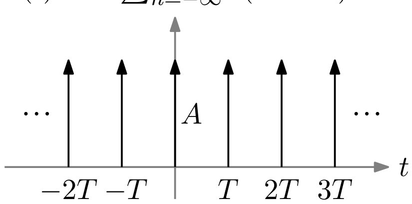
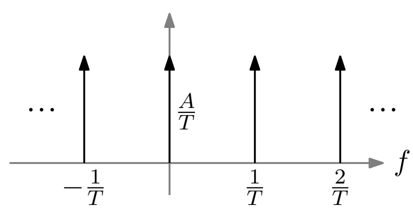
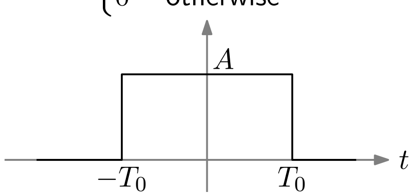
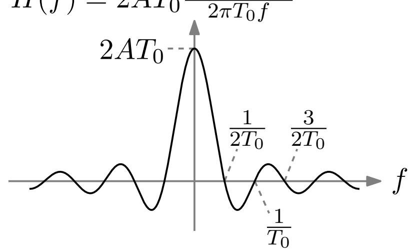
Inverse Fourier transform of $H(f)$:

$$\mathcal{F}^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,\mathrm{d}f = h(t)$$

Properties of the Fourier transform:

- Linearity:
$$\mathcal{F}\{\alpha x(t) + \beta y(t)\} = \alpha\mathcal{F}\{x(t)\} + \beta\mathcal{F}\{y(t)\}$$
- Symmetry:
$$\text{if } \mathcal{F}\{h(t)\} = H(f) \text{ then } \mathcal{F}\{H(t)\} = h(-f)$$
- Time shift:
$$\mathcal{F}\{x(t-t_0)\} = e^{-j2\pi ft_0}\mathcal{F}\{x(t)\}$$
- Time-frequency scaling:
$$\text{if } \mathcal{F}\{h(t)\} = H(f) \text{ then } \mathcal{F}\{h(\alpha t)\} = \left|\frac{1}{\alpha}\right| H\left(f/\alpha\right)$$
- Convolution:
  - Time-domain convolution corresponds to frequency-domain multiplication:
$$\mathcal{F}\{h(t) * x(t)\} = \mathcal{F}\{h(t)\} \cdot \mathcal{F}\{x(t)\}$$
  - Frequency-domain convolution corresponds to time-domain multiplication:
$$\mathcal{F}\{h(t) \cdot x(t)\} = \mathcal{F}\{h(t)\} * \mathcal{F}\{x(t)\}$$
- Even and odd functions:
  - If $h(t)$ is real, $H(f)$ has even real and odd imaginary parts
  - If $h(t)$ is real and even, $H(f)$ is also real and even:
$$\mathcal{F}\{h_e(t)\} = H_e(f) = \int_{-\infty}^{\infty} h_e(t)\cos(2\pi ft)\,\mathrm{d}t$$
  - If $h(t)$ is real and odd, $H(f)$ is imaginary and odd:
$$\mathcal{F}\{h_o(t)\} = H_o(f) = -j\int_{-\infty}^{\infty} h_o(t)\sin(2\pi ft)\,\mathrm{d}t$$

Fourier transform pairs:

Time domain	Frequency domain
$h(t) = A\cos(2\pi f_0 t)$ 	$H(f)$ Real 
$h(t) = A\sin(2\pi f_0 t)$ 	$H(f)$ Imaginary 
$h(t) = A\delta(t)$ 	$H(f)$ 
$h(t) = A\sum_{n=-\infty}^{-\infty}\delta(t-nT)$ 	$H(f)$ 
$h(t) = \begin{cases} A & \text{if } -T_0 < t < T_0 \\ 0 & \text{otherwise} \end{cases}$ 	$H(f) = 2AT_0\frac{\sin(2\pi T_0 f)}{2\pi T_0 f}$ 

Analog-to-digital conversion

Signal-to-quantisation-noise ratio:

$$\mathrm{SQNR} = 10\log_{10}\frac{P_v}{P_q}$$

For sinusoidal signal with amplitude $\alpha R/2$:

$$\mathrm{SQNR} = 6.02B + 20\log_{10}\alpha + 1.76$$

Discrete-time Fourier transform (DTFT)

DTFT:

$$X(f_\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f_\omega n}$$
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

IDTFT:

$$\begin{aligned} x[n] &= \int_{-1/2}^{1/2} X(f_\omega)e^{j2\pi f_\omega n}\,\mathrm{d}f_\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,\mathrm{d}\omega \end{aligned}$$

Properties of the DTFT

- Linearity:
$$\mathcal{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(\omega) + \beta X_2(\omega)$$
- Time shift:
$$\mathcal{F}\{x[n-k]\} = e^{-j\omega k}X(\omega)$$
- Frequency shift:
$$\mathcal{F}\{e^{j\omega_0 n}x[n]\} = X(\omega - \omega_0)$$
- Time:
$$\mathcal{F}\{x[-n]\} = X(-\omega)$$
- Conjugation:
$$\mathcal{F}\{x^*[n]\} = X^*(-\omega)$$
- Convolution:
$$\mathcal{F}\{x_1[n] * x_2[n]\} = X_1(\omega)\cdot X_2(\omega)$$

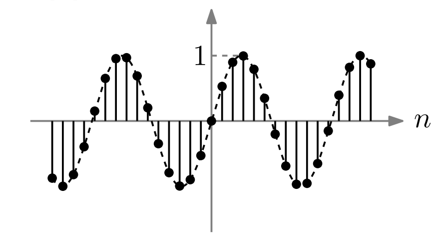
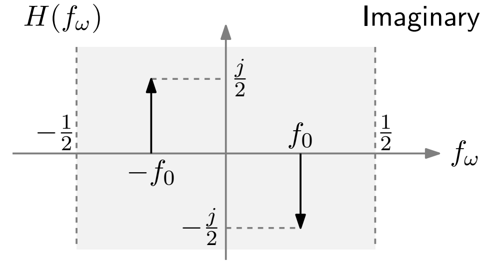
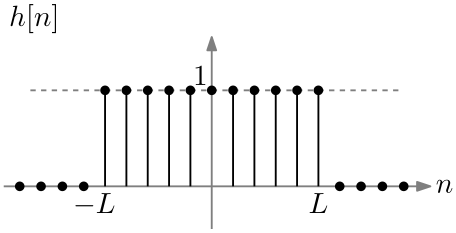
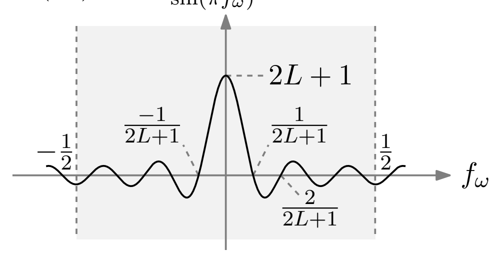
- Windowing:

$$\mathcal{F}\{x_1[n] \cdot x_2[n]\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda)\, \mathrm{d}\lambda$$

- Differentiation:

$$\mathcal{F}\{nx[-n]\} = j \frac{\partial}{\partial \omega} X(\omega)$$

DTFT transform pairs:

Discrete-time domain	Frequency domain
$h[n] = \sin(2\pi f_0 n)$ 	$H(f_\omega)$ Imaginary 
$h[n]$ 	$H(f_\omega) = \frac{\sin(\pi(2L+1)f_\omega)}{\sin(\pi f_\omega)}$ Real 

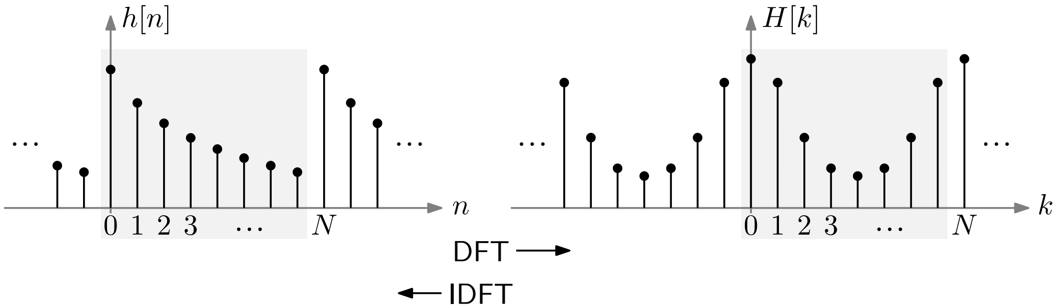
Discrete Fourier transform (DFT)

DFT of $h[n]$:

$$H[k] = \mathrm{DFT}\{h[n]\} = \sum_{n=0}^{N-1} h[n] e^{-j2\pi kn/N}$$

IDFT of $H[k]$:

$$h[n] = \mathrm{IDFT}\{H[k]\} = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi kn/N}$$



Properties of the DFT:

- Periodicity: Both $h[n]$ and $H[k]$ are periodic with same period N

$$h[n] = h[n + iN] \quad \text{and} \quad H[k] = H[k + iN] \qquad i \in \text{integers}$$

- Linearity:

$$\mathrm{DFT}\left\{\alpha x[n] + \beta y[n]\right\} = \alpha \mathrm{DFT}\{x[n]\} + \beta \mathrm{DFT}\{y[n]\}$$

- Symmetry:

$$\text{if } \mathrm{DFT}\{h[n]\} = H[k] \text{ then } \mathrm{DFT}\{H[n]\} = N \cdot h[-k] = N \cdot h[N-k]$$

- Even and odd time sequences:
  - If $h[n]$ is even, then $h[n] = h[-n] = h[N-n]$
  - If $h[n]$ is odd, then $h[n] = -h[-n] = -h[N-n]$
  - If $h[n]$ is real, $H[k]$ has an even real and an odd imaginary part
  - If $h[n]$ is real and even, $H[k]$ is also real and even:

$$\mathrm{DFT}\{h_e[n]\} = H_e[k] = \sum_{n=0}^{N-1} h_e[n] \cos(2\pi kn/N)$$

  - If $h[n]$ is real and odd, $H[k]$ is imaginary and odd:

$$\mathrm{DFT}\{h_o[n]\} = H_o[k] = -j \sum_{n=0}^{N-1} h_o[n] \sin(2\pi kn/N)$$

- Time reversal:

$$\mathrm{DFT}\{x[-n]\} = \mathrm{DFT}\{x[N-n]\} = X[N-k] = X[-k]$$

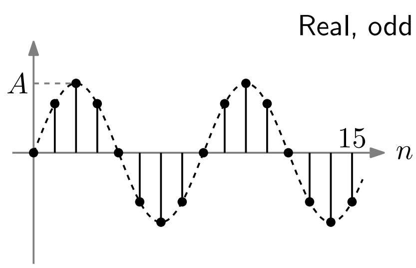
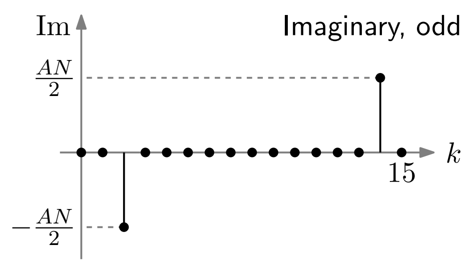
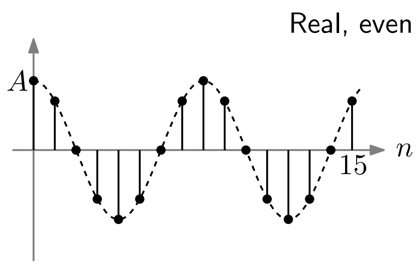
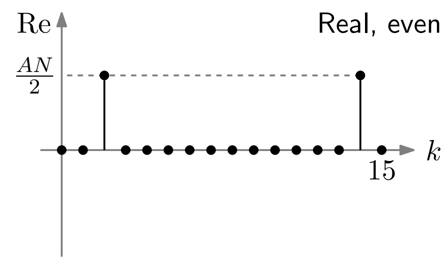
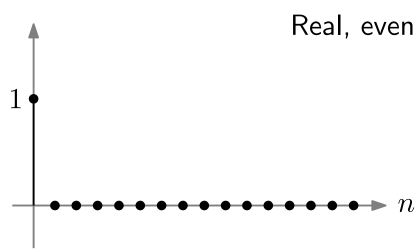
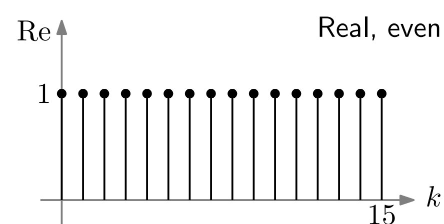
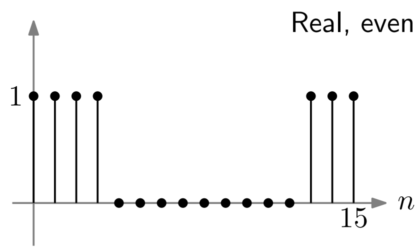
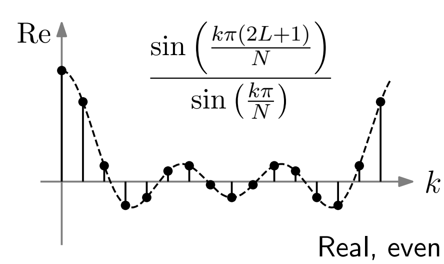
- Complex conjugate:

$$\mathrm{DFT}\{x^*[n]\} = X^*[N-k]$$

DFT identities:

- If $x[n] = x_1[n] + jx_2[n]$ then $X_1[k] = \frac{1}{2}X[k] + \frac{1}{2}X^*[N-k]$ and $X_2[k] = \frac{1}{2j}X[k] - \frac{1}{2j}X^*[N-k]$

DFT pairs:

Discrete-time domain	Frequency domain
	
	
	
	

Discrete convolution

Discrete convolution of $h[n]$ and $x[n]$:

$$h[n] * x[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

Circular convolution:

$$\begin{aligned} h[n] \underset{N}{\circledast} x[n] &= \sum_{i=0}^{N-1} h[i]\tilde{x}[n-i] \\ &= \sum_{i=0}^{N-1} x[i]\tilde{h}[n-i] \end{aligned}$$

Convolution and the DFT:

$$\begin{aligned} \mathrm{DFT}\{x[n]y[n]\} &= \frac{1}{N}\mathrm{DFT}\{x[n]\} \underset{N}{\circledast} \mathrm{DFT}\{y[n]\} \\ \mathrm{DFT}\{x[n] \underset{N}{\circledast} y[n]\} &= \mathrm{DFT}\{x[n]\} \cdot \mathrm{DFT}\{y[n]\} \end{aligned}$$

Discrete energy signals

Parseval:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, \mathrm{d}\omega$$

Power density spectrum: $S_{xx}(\omega) = |X(\omega)|^2$

Cross-correlation:

$$r_{xy}[i] = \sum_{n=-\infty}^{\infty} x[n]y[n-i] = \sum_{n=-\infty}^{\infty} x[n+i]y[n]$$

Cross-correlation and convolution: $x[i] * y[-i] = r_{xy}[i]$

Cross-correlation in frequency domain: $\mathcal{F}\{r_{xy}[i]\} = X(\omega)Y(-\omega)$

Cross-correlation symmetry: $r_{yx}[i] = r_{xy}[-i]$

Cross-correlation via DFT after zero padding: $\mathrm{DFT}\{r_{xy}[i]\} = X[k]\,Y^*[k]$

Autocorrelation:

$$r_{xx}[i] = \sum_{n=-\infty}^{\infty} x[n]x[n-i] = \sum_{n=-\infty}^{\infty} x[n+i]x[n]$$

Autocorrelation and energy: $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x$

Autocorrelation in frequency domain: $\mathcal{F}\{r_{xx}[i]\} = |X(\omega)|^2$

Autocorrelation via DFT after zero padding: $\mathrm{DFT}\{r_{xx}[i]\} = X[k]\,X^*[k] = |X[k]|^2$

Bounds:

$$|r_{xx}[i]| \le r_{xx}[0] = E_x$$
$$|r_{xy}[i]| \le \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{E_x E_y}$$

Discrete power signals

Parseval:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$

Power density spectrum: $S_{xx}[k] = \dfrac{1}{N}\,|X[k]|^2$

Cross-correlation:

$$r_{xy}[i] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]y[n-i]$$

Cross-correlation and convolution: $x[i] \underset{N}{\circledast} y[-i] = N r_{xy}[i]$

Cross-correlation in frequency domain: $N \cdot \mathrm{DFT}\{r_{xy}[i]\} = X[k]\,Y^*[k]$

Estimating cross-correlation from windows:

$$r_{xy}[i] \approx \frac{1}{|\mathcal{K}|} \sum_{n\in\mathcal{K}} x[n]y[n-i]$$

Autocorrelation:

$$r_{xx}[i] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]x[n-i]$$

Autocorrelation and power: $r_{xx}[0] = P_x$

Autocorrelation in frequency domain: $N \cdot \mathrm{DFT}\{r_{xx}[i]\} = N S_{xx}[k]$

Correlation of periodic signals with period N:

$$r_{xy}[i] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]y[n-i]$$
$$r_{xx}[i] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]x[n-i]$$

Bounds:

$$|r_{xx}[i]| \leq r_{xx}[0] = P_x$$
$$|r_{xy}[i]| \leq \sqrt{r_{xx}[0]r_{yy}[0]} = \sqrt{P_x P_y}$$

Discrete-time systems

Time-invariant system:

$$\begin{aligned} \text{if} \qquad & \mathcal{T}\{x[n]\} = y[n] \\ \text{then} \qquad & \mathcal{T}\{x[n-k]\} = y[n-k] \end{aligned}$$

Linear system: $\mathcal{T}\Big\{\sum_{i=1}^{N} \alpha_i x_i[n]\Big\} = \sum_{i=1}^{N} \alpha_i \mathcal{T}\{x_i[n]\}$

Linear time-invariant (LTI) system: $y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$

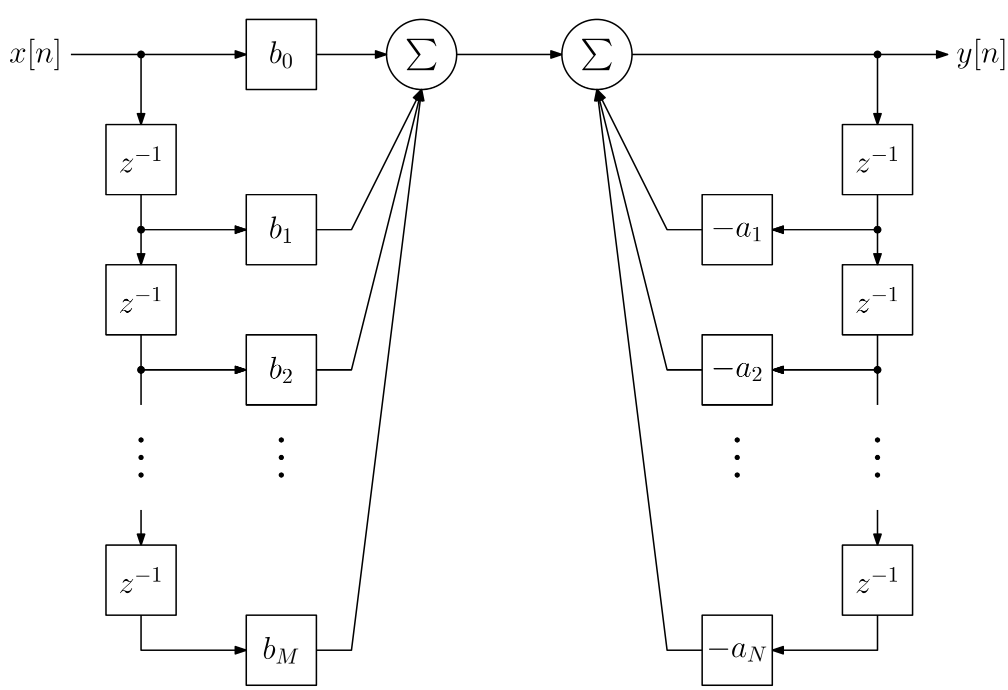
Causal LTI system: $h[i] = 0$ for all $i < 0$

BIBO-stable LTI system: $\sum_{i=-\infty}^{\infty} |h[i]| < \infty$

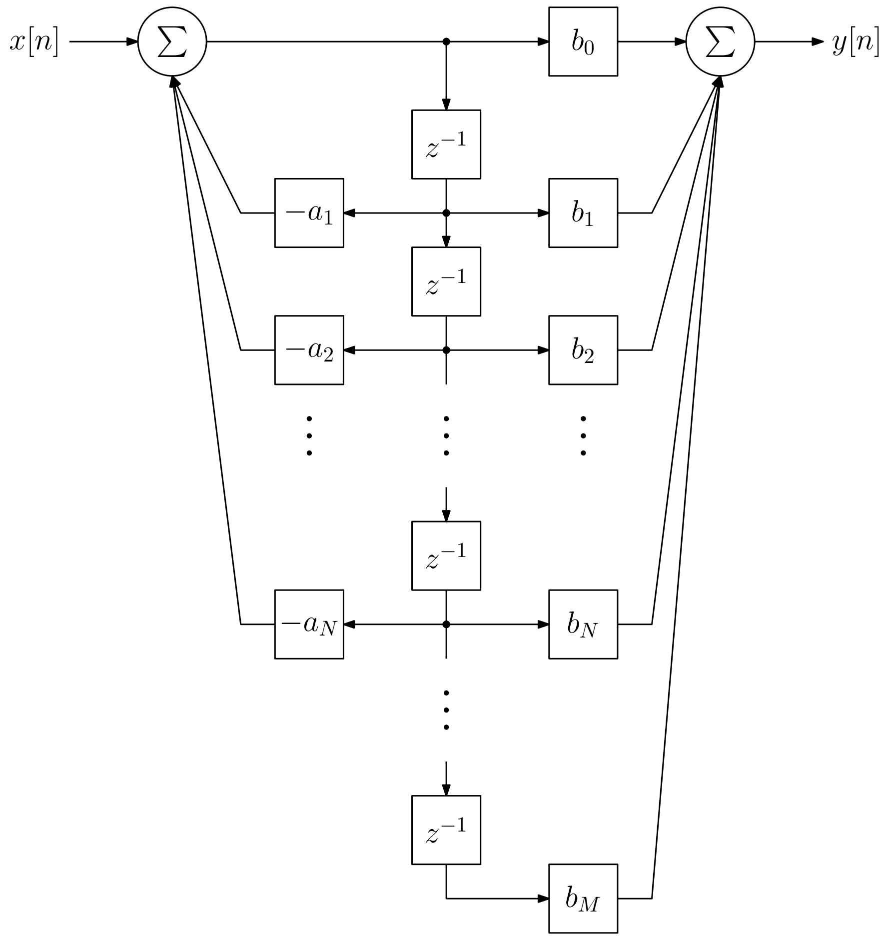
Linear constant-coefficient difference equation (LCCDE):

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

LCCDE direct form I:



LCCDE direct form II:



Overlap-and-add procedure:

- Choose a suitable block length L
- Zero pad $h[n]$ to length $N \geq L + P - 1$
- Calculate $H[k] = \text{FFT}\{h[n]\}$
- For each L-sample block of the input sequence:
  - Zero pad to length N
  - Calculate the FFT
  - Multiply with $H[k]$
  - Calculate the IFFT
  - Add to $y[n]$, overlapping the last $N - L$ samples
- Final result: $y[n]$

Cross-correlation between LTI system input and output: $r_{yx}[i] = h[i] * r_{xx}[i]$

The z-transform

The z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Properties of the z-transform:

- Linearity:
$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha \mathcal{Z}\{x[n]\} + \beta \mathcal{Z}\{y[n]\}$$

- Time shift:
$$\mathcal{Z}\{x[n-k]\} = z^{-k} \mathcal{Z}\{x[n]\}$$

- Time reversal:
$$\text{if } \mathcal{Z}\{x[n]\} = X(z) \text{ then } \mathcal{Z}\{x[-n]\} = X(1/z)$$

- Convolution:
$$\mathcal{Z}\{x[n] * y[n]\} = \mathcal{Z}\{x[n]\} \cdot \mathcal{Z}\{y[n]\}$$

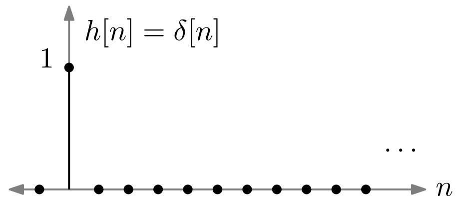
- Correlation:
$$\text{if } \mathcal{Z}\{x[n]\} = X(z) \text{ and } \mathcal{Z}\{y[n]\} = Y(z) \text{ then } \mathcal{Z}\{r_{xy}[i]\} = X(z)Y(z^{-1})$$

- Initial value theorem:
$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{z\to\infty} X(z) = x[0]$$

- Final value theorem:
$$\text{if } x[n] = 0 \text{ for } n < 0 \text{ then } \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)X(z)$$

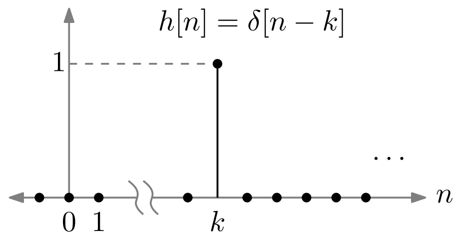
Pairs of z-transforms:

Discrete time-domain ⇔ z-transform



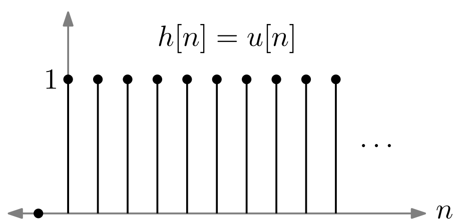
$$H(z) = 1$$

all z



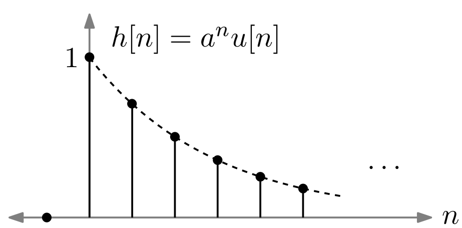
$$H(z) = z^{-k}$$

$z \neq 0$
$z \neq \infty$



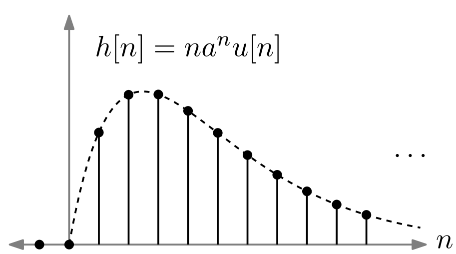
$$H(z) = \frac{1}{1 - z^{-1}}$$

$|z| > 1$



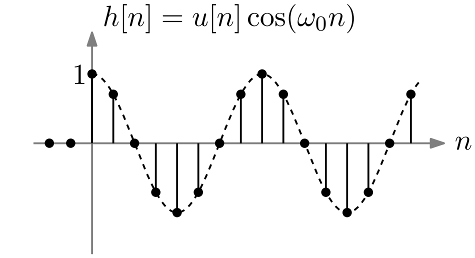
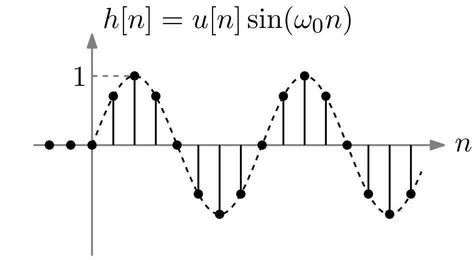
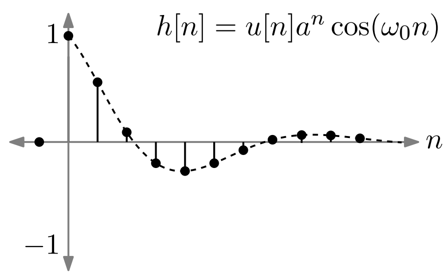
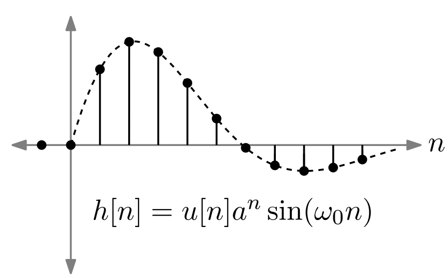
$$H(z) = \frac{1}{1 - az^{-1}}$$

$|z| > |a|$



$$H(z) = \frac{az^{-1}}{(1 - az^{-1})^2}$$

$|z| > |a|$

Discrete time-domain ⇔ z-transform	
	$H(z) = \dfrac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$ $\lvert z\rvert > 1$
	$H(z) = \dfrac{(\sin\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$ $\lvert z\rvert > 1$
	$H(z) = \dfrac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$ $\lvert z\rvert > \lvert a\rvert$
	$H(z) = \dfrac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$ $\lvert z\rvert > \lvert a\rvert$

Transfer function of LCCDE system:

$$H(z) = z^{N-M}\frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}}$$

Partial fraction expansion steps:

1. If $M \geq N$, use long division to get to $M < N$ (do this with powers of z^{-1})
2. Convert equation to have positive powers of z
3. Factorise $X(z)/z$
4. Do partial fraction expansion
5. Convert back to powers of z^{-1}
6. Inverse by inspection using known z-transform pairs

Complex conjugate poles:

$$2u[n] \cdot |A| \cdot |p|^n \cos(\angle p \cdot n + \angle A) \qquad \Leftrightarrow \qquad \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

Discrete filters

All-pass filter:

$$\begin{aligned} H(z) &= \frac{z^{-N} + a_{N-1} z^{-(N-1)} + a_{N-2} z^{-(N-2)} + \ldots + a_1 z^{-1} + a_0}{1 + a_{N-1} z^{-1} + a_{N-2} z^{-2} + \ldots + a_1 z^{-(N-1)} + a_0 z^{-N}} \\ &= \prod_{k=0}^{N_r} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=0}^{N_c} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})} \end{aligned}$$

Bilinear transform ($K > 0$):

$$\begin{aligned} s &= \frac{1}{K} \frac{1 - z^{-1}}{1 + z^{-1}} \\ z &= \frac{1 + Ks}{1 - Ks} \end{aligned}$$

Butterworth LPF of order N has magnitude response:

$$|H_c(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

with N poles at

$$s_k = \Omega_c e^{j\pi \frac{N+1+2k}{2N}} \text{ for } k = 0, 1, \ldots, N-1$$

Continuous to discrete filter design procedure:

1. Specification in discrete time
2. Pre-warp specification frequencies $\Omega = \frac{1}{K} \tan\left(\frac{\omega}{2}\right)$
3. Design continuous-time filter to pre-warped specification
4. Substitute bilinear transform $s = \frac{1}{K} \frac{1 - z^{-1}}{1 + z^{-1}}$

Filter transforms:

- Low-pass to low-pass transform:

$$g(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$
$$\alpha = \frac{\sin\left(\frac{\omega_p - \omega_d}{2}\right)}{\sin\left(\frac{\omega_p + \omega_d}{2}\right)}$$

- Low-pass to high-pass transform:

$$g(z^{-1}) = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$
$$\alpha = -\frac{\cos\left(\frac{\omega_p + \omega_d}{2}\right)}{\cos\left(\frac{\omega_p - \omega_d}{2}\right)}$$

- Low-pass to band-pass transform:

$$g(z^{-1}) = -\frac{\frac{k-1}{k+1} - \frac{2\alpha k}{k+1} z^{-1} + z^{-2}}{1 - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1} z^{-2}}$$
$$\alpha = \frac{\cos\left(\frac{\omega_{du} + \omega_{dl}}{2}\right)}{\cos\left(\frac{\omega_{du} - \omega_{dl}}{2}\right)}$$
$$k = \tan\left(\frac{\omega_p}{2}\right) / \tan\left(\frac{\omega_{du} - \omega_{dl}}{2}\right)$$

- Low-pass to band-stop transform:

$$g(z^{-1}) = -\frac{\frac{1-k}{1+k} - \frac{2\alpha}{k+1} z^{-1} + z^{-2}}{1 - \frac{2\alpha}{k+1} z^{-1} + \frac{1-k}{1+k} z^{-2}}$$
$$\alpha = \frac{\cos\left(\frac{\omega_{du} + \omega_{dl}}{2}\right)}{\cos\left(\frac{\omega_{du} - \omega_{dl}}{2}\right)}$$
$$k = \tan\left(\frac{\omega_p}{2}\right) \tan\left(\frac{\omega_{du} - \omega_{dl}}{2}\right)$$