Linear regression

Polynomial regression and basis functions

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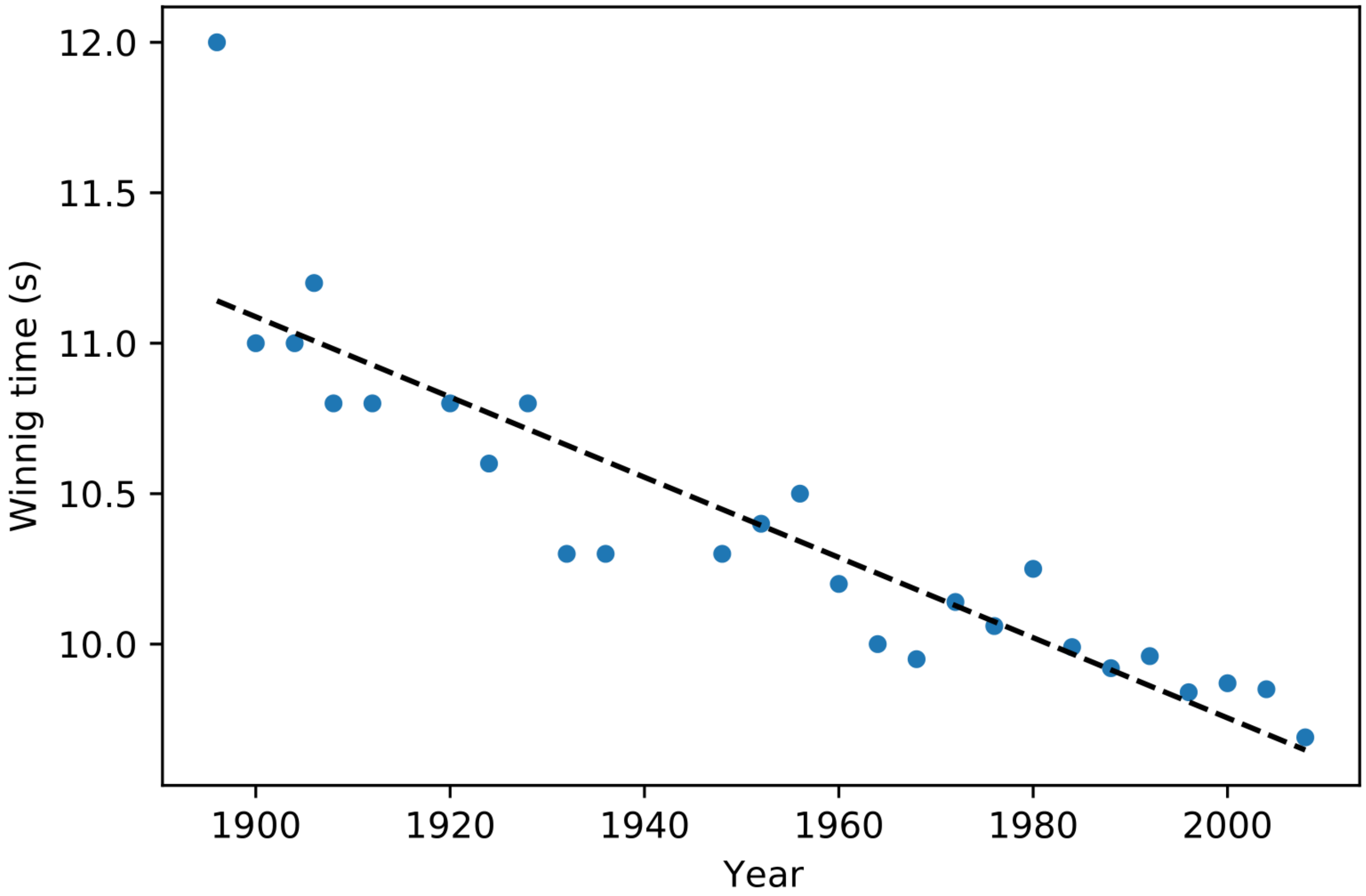
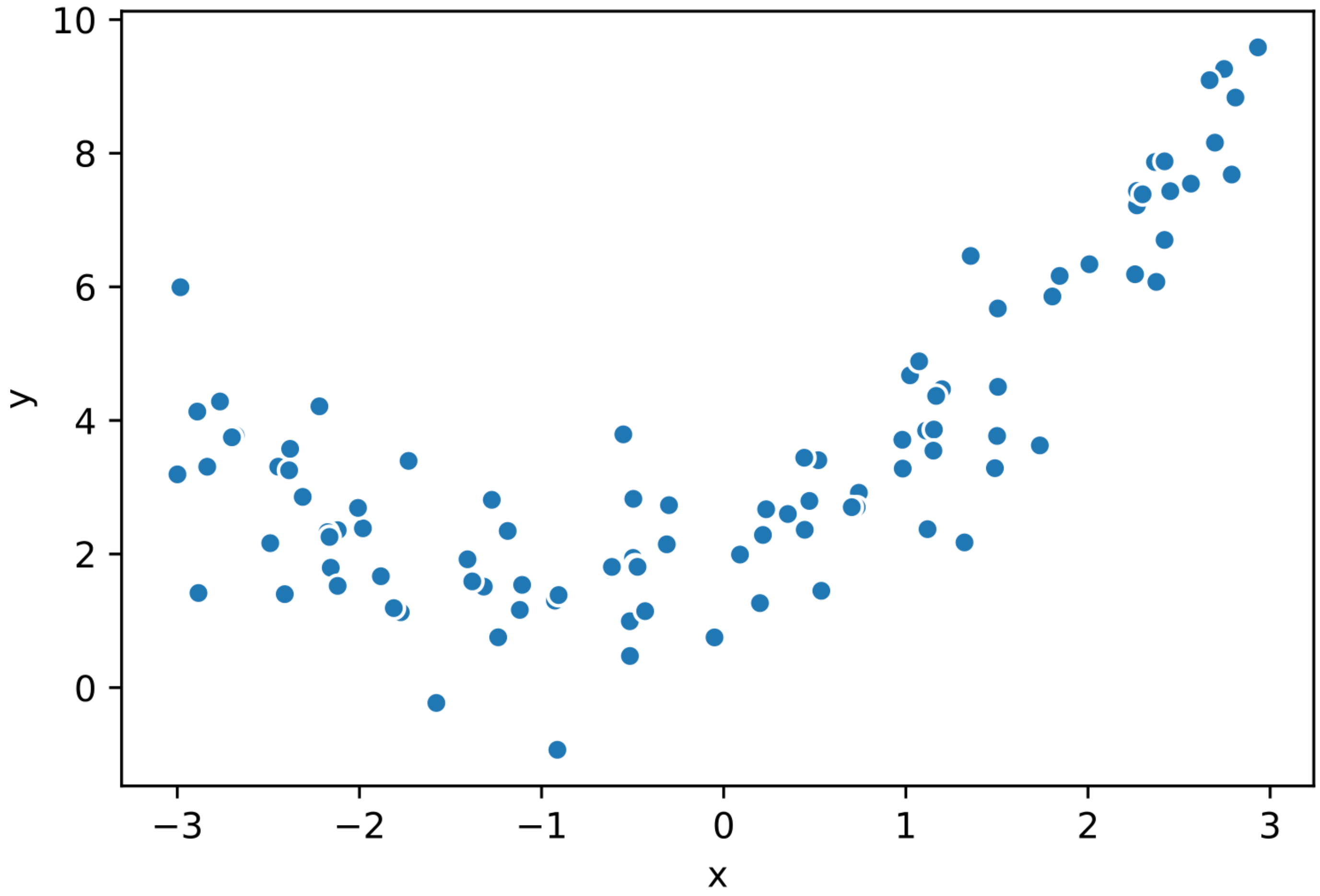
Linear regression



Non-linear relationship



Polynomial regression

Multiple linear regression recap:

$$f(\underline{x}; \underline{w}) = w_0 + w_1 x_1 + \dots + w_D x_D = \underline{w}^T \underline{x}$$

Fit on data $\{(\underline{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ using:

$$J(\underline{w}) = \sum_{n=1}^{N} \left(y^{(n)} - f(\underline{x}^{(n)}; \underline{w})\right)^2$$

$$= (\underline{y} - \underline{X}\,\underline{w})^T (\underline{y} - \underline{X}\,\underline{w}), \text{ with}$$

$$\underline{X} = \begin{bmatrix} — (\underline{x}^{(1)})^T — \\ — (\underline{x}^{(2)})^T — \\ \vdots \\ — (\underline{x}^{(N)})^T — \end{bmatrix} ; \quad \underline{y} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(N)} \end{bmatrix}$$

Design matrix

Solution: $\hat{\underline{w}} = (\underline{X}^T \underline{X})^{-1} \underline{X}^T \underline{y}$ Normal equations

Polynomial regression:

What do we do if we want to fit

$$f(x; w_0, w_1, w_2) = w_0 + w_1 x + w_2 x^2 \ ?$$

Let's define

$$\underline{\phi}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

We can then write $f(x; \underline{w}) = \underline{w}^T \underline{\phi}(x)$ "$\underline{w}^T \underline{x}$"

Now we can solve the problem exactly as for multiple linear regression by "pretending" that $\underline{\phi}(x)$ is $\underline{x}$.

Our design matrix would now become:

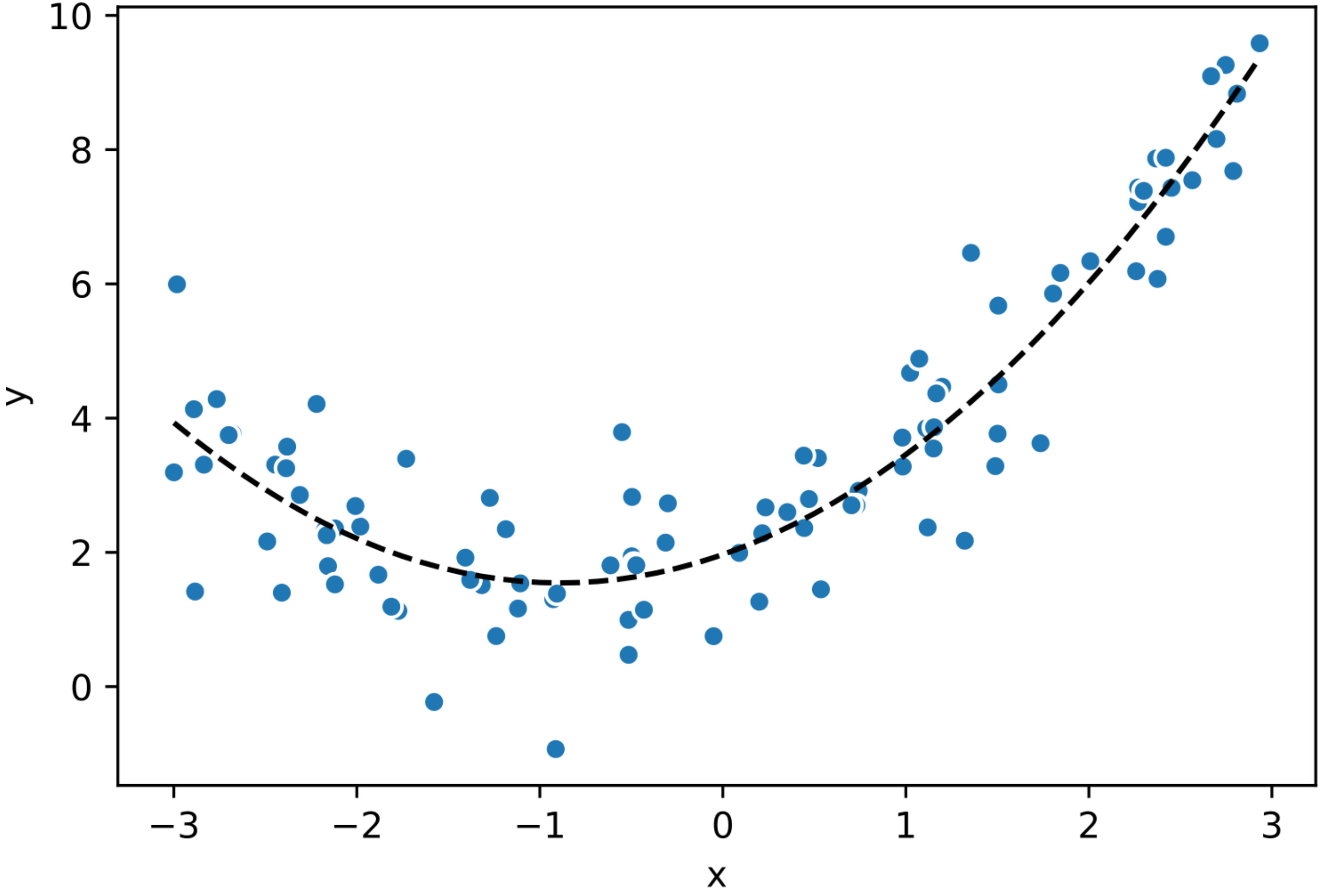
$$\underline{\Phi} = \begin{bmatrix} — \underline{\phi}(x^{(1)})^T — \\ — \phi(x^{(2)})^T — \\ \vdots \\ — \phi(x^{(N)})^T — \end{bmatrix} = \begin{bmatrix} 1 & x^{(1)} & (x^{(1)})^2 \\ 1 & x^{(2)} & (x^{(2)})^2 \\ & \vdots & \\ 1 & x^{(N)} & (x^{(N)})^2 \end{bmatrix}$$

Other example:

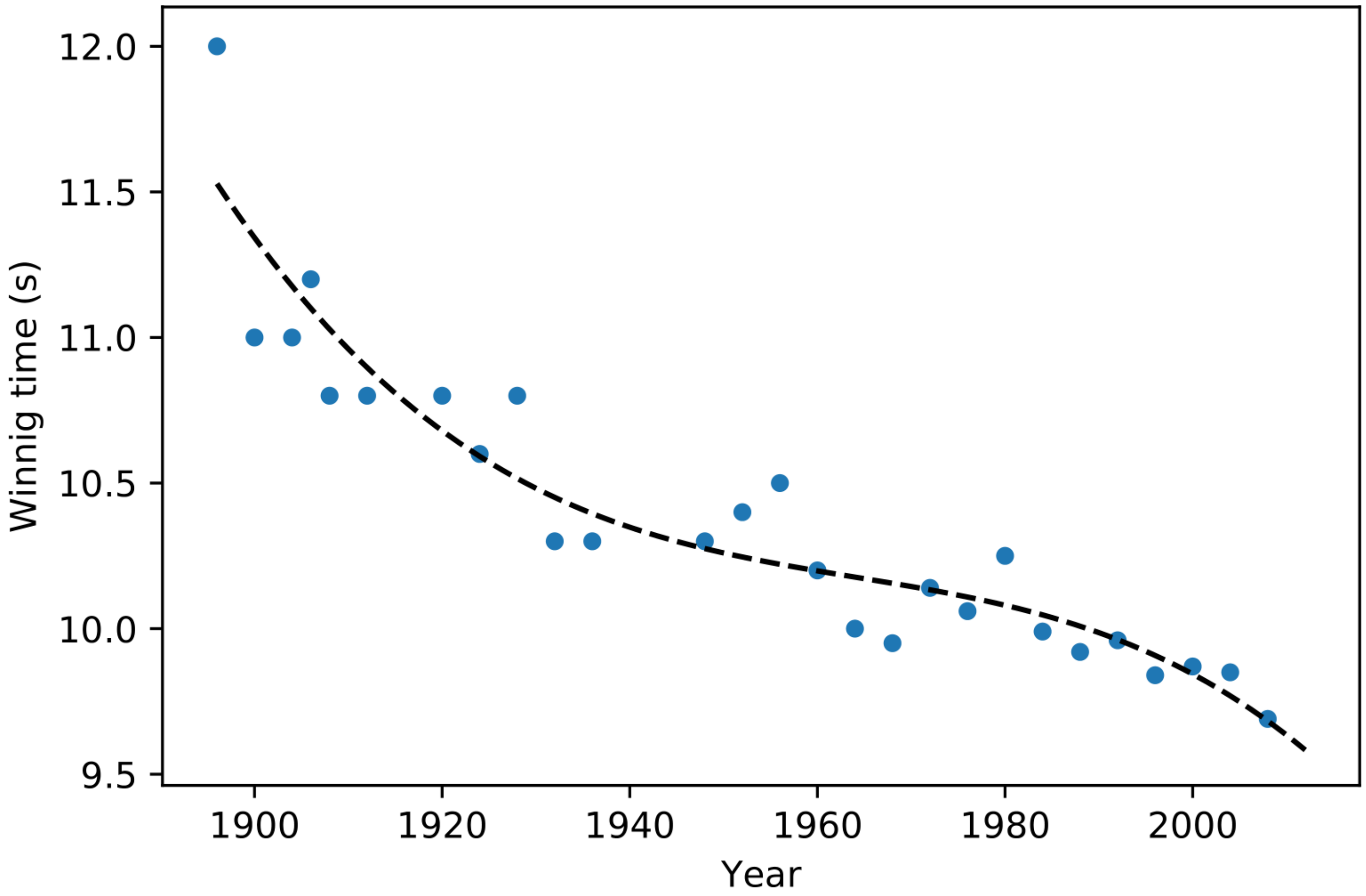
$$f(\underline{x}; \underline{w}) = w_0 + w_1 x_1 + w_2 x_2 + w_3 x_1 x_2 + w_4 x_1^2 + w_5 x_2^2$$

with $\underline{\phi}(\underline{x}) = [1 \quad x_1 \quad x_2 \quad x_1 x_2 \quad x_1^2 \quad x_2^2]^T$

Polynomial regression



3rd-order polynomial regression



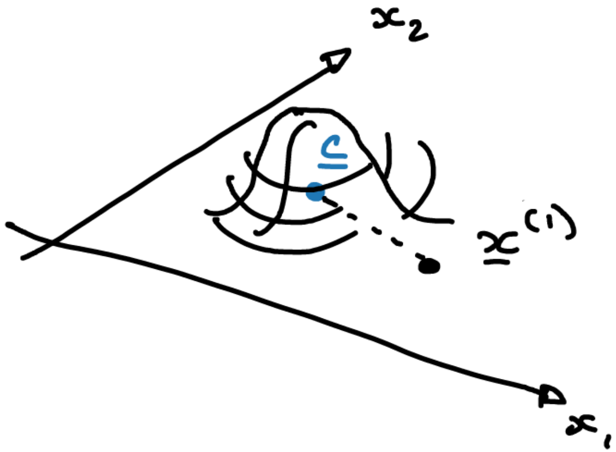
Basis functions

Instead of just polynomials, we can actually put any function in

$$\underline{\phi}(\underline{x}) = [\phi_1(\underline{x}) \quad \phi_2(\underline{x}) \quad \cdots \quad \phi_K(\underline{x})]^T$$

e.g. sin, cos, log, exp, FFT, etc.
This can be quite useful if we have some inside domain knowledge of the data and problem.

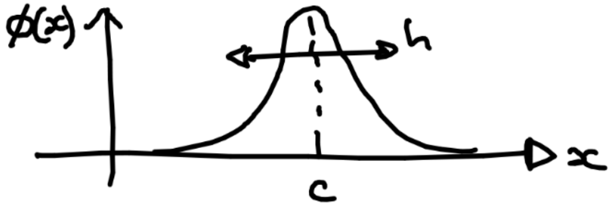
RBF in 2 dimensions:



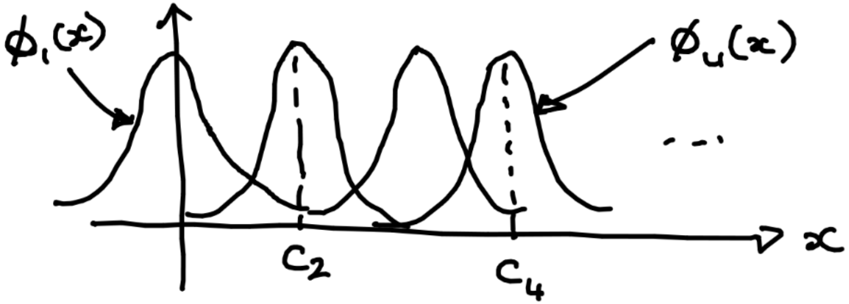
One possible choice as a basis function is the radial basis function (RBF):

$$\phi(\underline{x}) = \exp\left\{ -\frac{(\underline{x}-\underline{c})^T(\underline{x}-\underline{c})}{h^2} \right\}$$

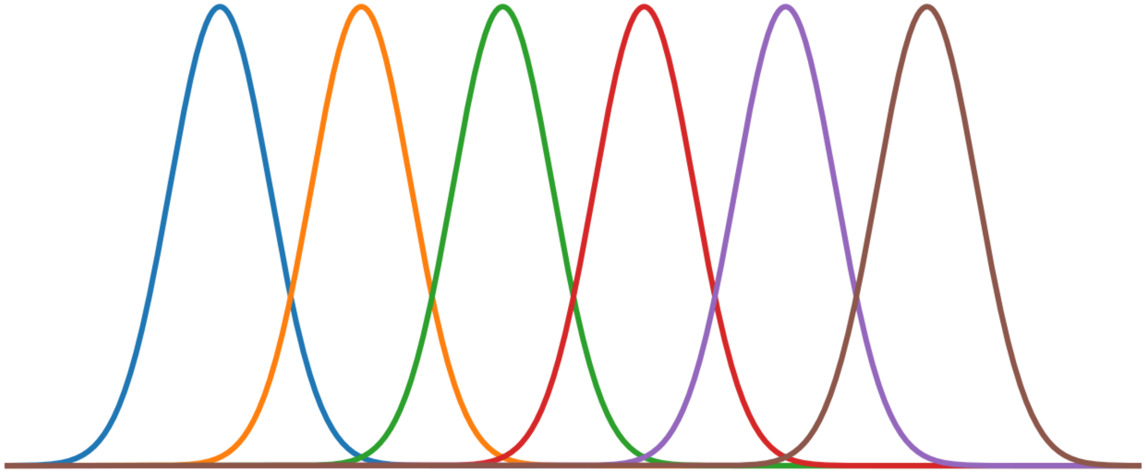
In the 1-D case:



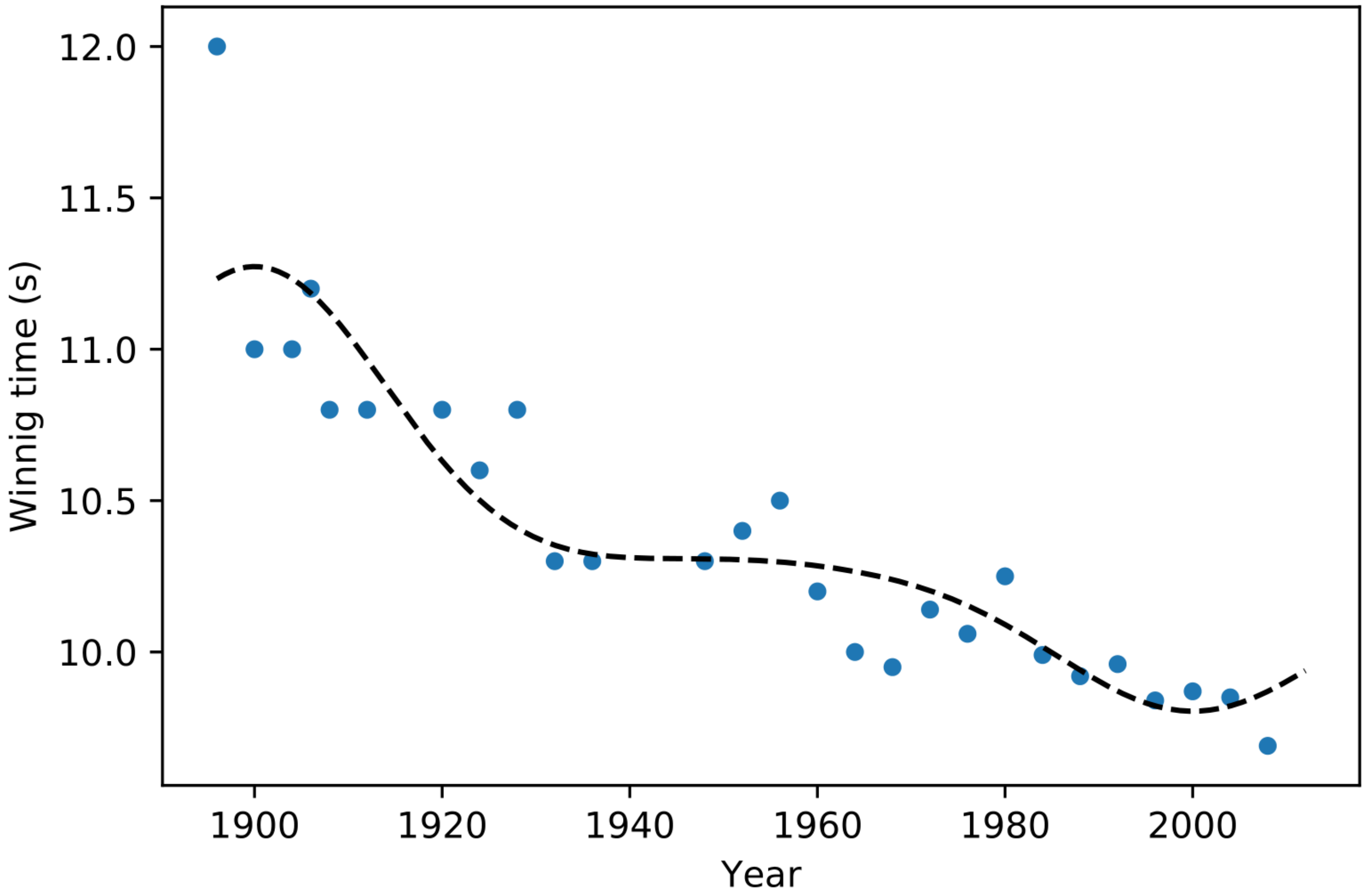
Can even have a family of RBFs:



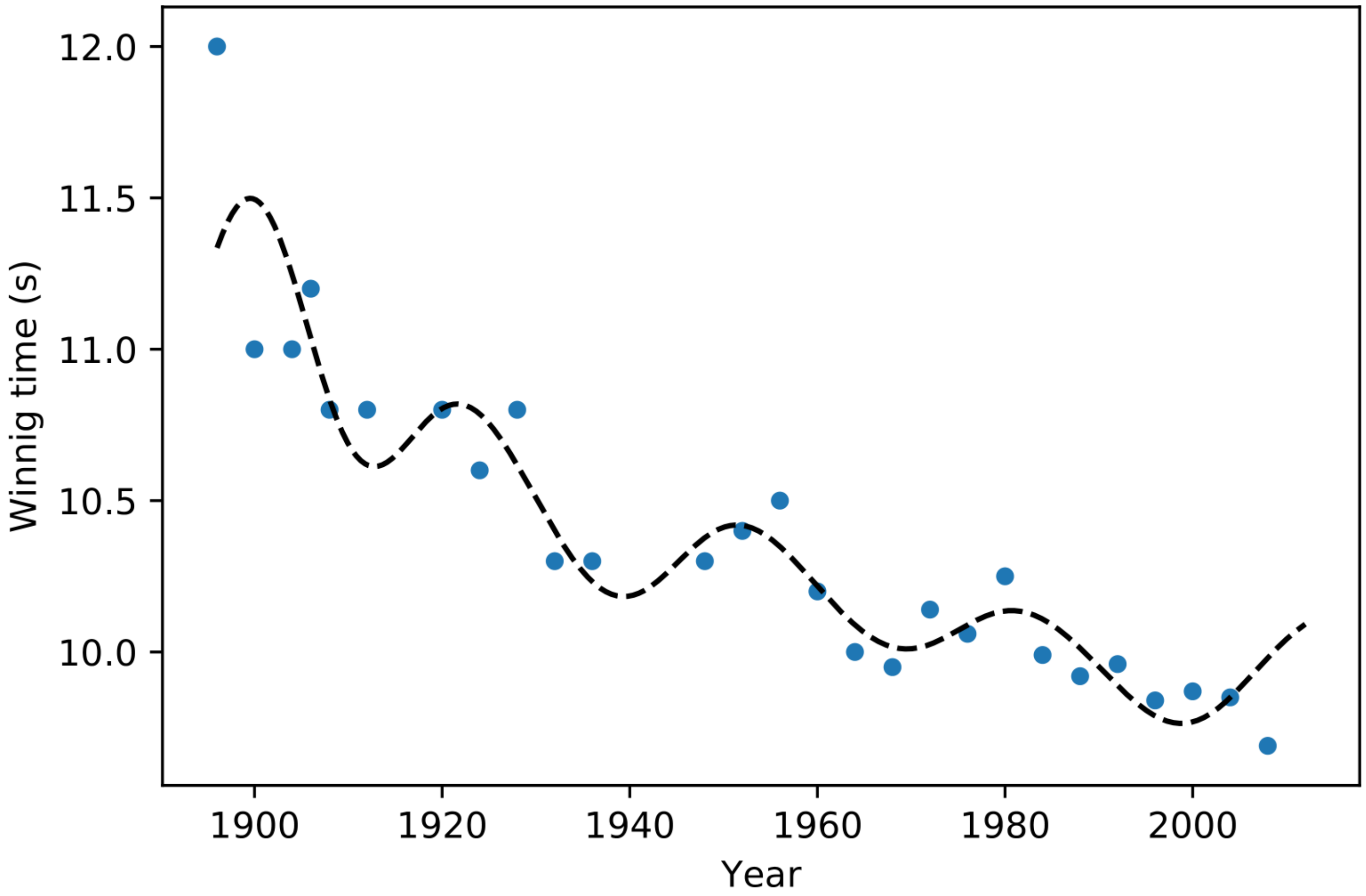
Basis functions: RBF

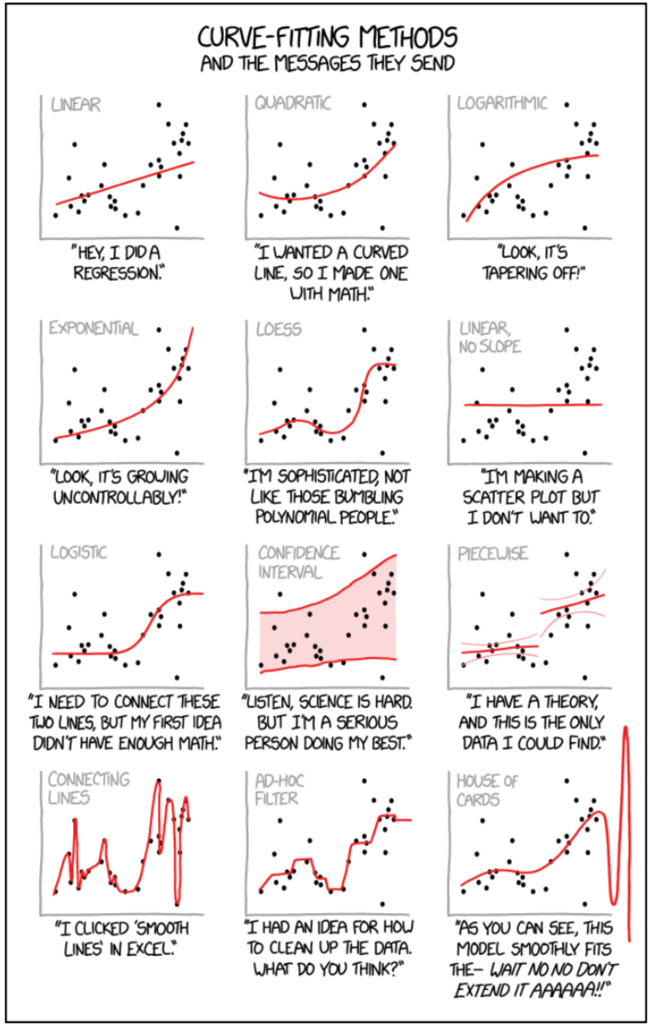


RBF with $c = [1900, 1950, 2000]$ and $h = 20$



RBF with $c = [1900, 1910, \ldots, 2000]$ and $h = 10$





Source: https://xkcd.com/2048/